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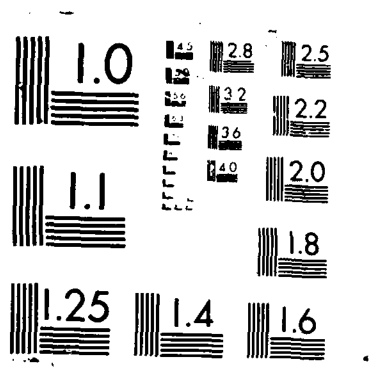
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Department of Statistics
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Chapel Hill, North Carolina

STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY
NUCLEAR SPACE VALUED MARTINGALES

by

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and

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STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY NUCLEAR SPACE VALUED MARTINGALES

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Abstract

The paper presents a theory of stochastic evolution equations for nuclear space valued processes and provides a unified treatment of several examples from the field of applications. $(C_0,1)$ reversed evolution systems on countably Hilbertian nuclear spaces are also investigated.

Keywords: nuclear space, stochastic evolution equation, time dependent evolution, $(C_0,1)$-semigroup, interacting diffusion system, infinite dimensional process.

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INTRODUCTION

The aim of this paper is to present a theory of stochastic evolution equations governing processes that take values in duals of countably Hilbertian nuclear spaces. For Hilbert or Banach space valued processes such studies are available, e.g. in Curtain and Pritchard [3] and Kotelenez and Curtain [9].

In recent work arising from problems in such diverse fields as chemical kinetics, n-particle diffusions and neurophysiology, one comes up with a situation where a sequence of stochastic processes of interest converges weakly to an ∞-dimensional process satisfying a stochastic evolution equation on a suitable space of distributions. It is, therefore, of interest to develop a general theory of the existence and uniqueness of solutions of stochastic evolution equations in a dual Φ' of a nuclear space Φ where the driving force is a Φ'-valued martingale. We do this in Section 2 where we obtain an "evolution" or "mild solution" in the form of a stochastic integral with respect to the given (Φ'-valued) martingale. A feature of these equations is that in general, over any finite interval $[0,T]$, the solution lives in a Hilbert space Φ'_{p_T} while, when no finite interval is specified it can only be asserted that it takes values in Φ'.

We have to preface the stochastic part of our work by a study of deterministic evolution systems (including perturbed systems) defined on countably Hilbertian nuclear spaces. In applications, one is led to consider evolution systems on Φ' in the following manner. Initially the problem is defined on a Hilbert space H with a family of infinitesimal generators $\{\tilde{A}(t)\}$. It is often the case that we can find a Gelfand triplet $\Phi \hookrightarrow H \hookrightarrow \Phi'$ (with Φ a countably Hilbertian nuclear space) such that the restriction $A(t)$ of $\tilde{A}(t)$ on Φ is a continuous linear operator on Φ. Thus one is led to the question of when the family $\{A(t)\}$ generates an evolution system on Φ or, equivalently on Φ'. This is the problem considered in Section 1. The theory of evolution systems on Hilbert and Banach spaces has been developed extensively by a number of authors. They are constructed from (unbounded) infinitesimal generators $\{\tilde{A}(t)\}$ of C_0-semigroups. Evolution systems over locally convex spaces have been constructed from C_0-equicontinuous semigroups by K. Yosida [19] and from quasi-equicontinuous C_0-semigroups by Y.H. Choe [21].

We have not been able to find results of sufficient generality that we could use, viz., results on evolution systems on nuclear spaces constructed from $(C_0,1)$-semigroups. We derive these systems using the ideas from Kato's theory of evolution equations on Banach spaces (see Pazy [16], Chapter 5). A key notion in this construction is that of a stable family of generators or semigroups on Φ.

In the last section, the theory developed in Section 2 is applied to examples arising in various fields of applications. The stochastic equations of Hitsuda and Mitoma, Kallianpur and Wolpert, Kotelenez, and Mitoma [4,7,8,14] are shown to be particular cases of the equations of Section 2 and so the existence of a unique solution follows as a consequence of Theorem 2.1. It also follows that all the examples possess a family of stable generators. Mitoma's example in [14] is particularly interesting in that the evolution system is generated by a stable family of $(C_0,1)$-semigroups which are not equicontinuous.

1. EVOLUTION AND REVERSED EVOLUTION OPERATORS

Let Φ be a countably Hilbertian nuclear space whose topology τ is defined by an increasing sequence of Hilbertian norms $|\cdot|_1 \le |\cdot|_2 \le \dots \le |\cdot|_n \le \dots$. Let Φ_n be the completion of Φ by $|\cdot|_n$, Φ'_n the topological dual of Φ_n, $|\cdot|_{-n}$ the dual norm of Φ'_n and Φ' be the strong topological dual of Φ. The completeness of Φ implies

$$\Phi = \bigcap_{n=1}^{\infty} \Phi_n \quad \text{and} \quad \Phi' = \bigcup_{n=1}^{\infty} \Phi'_n.$$

We denote by $L(\Phi,\Phi)$ (respectively $L(\Phi',\Phi')$) the class of continuous linear operators from Φ to Φ (Φ' to Φ').

A two parameter family of operators $\{U(t,s) : 0 \le s \le t < \infty\}$ in $L(\Phi',\Phi')$ is said to be an *evolution system* on Φ' if the following two conditions are satisfied:

(i) $U(t,t) = I$, $U(t,r)U(r,s) = U(t,s)$ $\quad 0 \le s \le r \le t$,

(ii) For each $\psi \in \Phi'$ the map $(s,t) \to U(t,s)\psi$ is strongly continuous. We recall that for Φ and Φ' strong and weak convergence of sequences coincide.

Let $\{A'(t)\}_{t \ge 0}$ be a family in $L(\Phi',\Phi')$. We say that this family of operators generates the evolution system $\{U(t,s) : 0 \le s \le t < \infty\}$ if the following relations are satisfied:

$$\frac{d}{dt}U(t,s)\psi = A'(t)U(t,s)\psi \text{ for all } \psi \in \Phi' \quad 0 \le s \le t$$

$$\frac{d}{ds}U(t,s)\psi = -U(t,s)A'(s)\psi \text{ for all } \psi \in \Phi' \quad 0 \le s \le t.$$

For $s \le t$ define the operator $T(s,t) : \Phi \to \Phi$ by the relation

(1.1) $\quad (U(t,s)\psi)[\phi] = \psi[T(s,t)\phi]$ for all $\phi \in \Phi$, $\psi \in \Phi'$.

In a similar way for each $t \geq 0$ define the continuous linear operator $A(t): \Phi \to \Phi$ by

(1.2) $\quad (A'(t)\psi)[\phi] = \psi[A(t)\phi]$ for all $\phi \in \Phi$, $\psi \in \Phi'$.

Then it is not difficult to verify that the continuous linear operators $\{T(s,t): 0 \leq s \leq t < \infty\}$ on Φ have the following properties:

(1.3) $\quad T(s,t) = T(s,r)T(r,t) \quad 0 \leq s \leq r \leq t, \; T(t,t) = I.$

(1.4) $\quad$ For each $\phi \in \Phi$ the map $(s,t) \to T(s,t)\phi$ is Φ-continuous.

(1.5) $\quad \frac{d}{dt}T(s,t)\phi = T(s,t)A(t)\phi$ for all $\phi \in \Phi \quad 0 \leq s \leq t.$

(1.6) $\quad \frac{d}{ds}T(s,t)\phi = -A(s)T(s,t)\phi$ for all $\phi \in \Phi \quad 0 \leq s \leq t.$

Definition 1.1. A two parameter family of operators $T(s,t)$ $0 \leq s \leq t$ in $L(\Phi,\Phi)$ is said to be a *reversed evolution system* if it satisfies (1.3) and (1.4) above. If a family $\{A(t)\}_{t \geq 0}$ of linear operators on Φ satisfies equations (1.5) and (1.6) we say that $\{A(t)\}_{t \geq 0}$ generates the reversed evolution system $T(s,t)$. Relations (1.5) and (1.6) are called the *forward and backward equations*.

The main result of this section is Theorem 1.3 below where we give sufficient conditions on a family of linear operators on Φ to generate a reversed evolution system $\{T(s,t): 0 \leq s \leq t < \infty\}$ on Φ. Using the relations (1.1) and (1.2) we then have that the family $\{A'(t)\}_{t \geq 0}$ of linear operators on Φ' generates an evolution system $\{U(t,s): 0 \leq s \leq t < \infty\}$ on Φ'. It will be convenient to denote $U(t,s)$ by $T'(t,s)$ and refer to it as the adjoint of $T(s,t)$. This is particularly convenient when $T(s,t)$ is the primary object in our discussion.

Our results and examples on this work deal with semigroups of linear operators on Φ which are not necessarily equicontinuous as those presented in Yosida [20] for locally convex spaces or Miyadera [15] for Fréchet spaces. We have to deal with semigroups of linear operators of $(C_0,1)$-class defined below. The terminology is due to Babalola who has studied such semigroups on locally convex spaces [1].

Definition 1.2. A family $\{S(s) : s \geq 0\}$ of linear operators on Φ is said to be a $(C_0,1)$-*semigroup* if the following three conditions are satisfied:

(1) $S(s_1)S(s_2) = S(s_1 + s_2)$ for all $s_1, s_2 \geq 0$, $S(0) = I$.

(2) The map $s \to S(s)\phi$ is Φ-continuous for each $\phi \in \Phi$.

(3) For each $q \geq 0$ there exist numbers M_q, σ_q and $p \geq q$ such that

$$|S(s)\phi|_q \leq M_q e^{\sigma_q s} |\phi|_p \quad \text{for all } \phi \in \Phi,\ s \geq 0.$$

We recall that a semigroup $S(s)$ is called a C_0-semigroup if it satisfies (2) above. It is said to be an *equicontinuous* semigroup if it satisfies (1)-(2) and (3) with $\sigma_q = 0$, $q \geq 0$. Thus equicontinuous semigroups are special types of $(C_0,1)$-semigroups. The case of $\sigma_q = \sigma$, $q \geq 0$ is considered in [15] and [21].

The next two theorems characterize $(C_0,1)$-semigroups. Before presenting them we introduce some notation: Let $\{S(s) : s \geq 0\}$ be a $(C_0,1)$-semigroup on Φ. The infinitesimal generator A of $S(s)$ is defined as

$$A\phi = \lim_{s \downarrow 0} \frac{S(s)\phi - \phi}{s} \quad \text{(limit in } \Phi)$$

whenever the limit exists, the domain of A being the set $\mathcal{D}(A) \subset \Phi$ for which the above limit exists.

Let $\{\|\cdot\|_n : n \geq 0\}$ be any sequence of increasing norms on Φ also defining the τ-topology of Φ. Such a sequence of norms will henceforth be called τ-*compatible*. We will denote by $\Phi_{|n|}$ the $\|\cdot\|_n$-completion of Φ. Then $\Phi \subset \Phi_{|n|} \subset \Phi_{|m|}$ for $n \geq m$ and $\Phi = \cap_{n=0}^{\infty} \Phi_{|n|}$. Suppose that $A : \mathcal{D}(A) \subset \Phi \to \Phi$ is a densely defined closed linear operator. If for some $n \geq 0$ the linear operator

$$A : \mathcal{D}(A) \subset \Phi_{|n|} \to \Phi \subset \Phi_{|n|}$$

is closable in $\Phi_{|n|}$, then we denote by A_n the closure of A in $\Phi_{|n|}$.

The proofs of the following two results involve standard arguments and are therefore omitted.

<u>Theorem 1.1</u> (a). A C_0-semigroup $\{S(s): s \geq 0\}$ on Φ is a $(C_0,1)$-semigroup if and only if there exists a sequence of τ-compatible norms $\{\|\cdot\|_n : n \geq 0\}$ on Φ and sequences of nonnegative numbers $\{\sigma_n\}_{n \geq 0}$ such that for each $n \geq 0$

$$\|S(s)\phi\|_n \leq e^{\sigma_n s} \|\phi\|_n \text{ for all } \phi \in \Phi,\ s \geq 0. \tag{1.7}$$

(b). If $\{S(s): s \geq 0\}$ is a $(C_0,1)$-semigroup on Φ then there exists a family of Banach spaces $\{\Phi_{|n|} : n \geq 0\}$ whose norms $\{\|\cdot\|_n : n \geq 0\}$ are τ-compatible, such that for each $n \geq 0$ $S(s)$ can be extended to a C_0-semigroup $\{S^n(s): s \geq 0\}$ of linear operators on $\Phi_{|n|}$.

Theorem 1.2. A necessary and sufficient condition for a closed linear operator A on Φ to be the infinitesimal generator of a unique $(C_0,1)$-semigroup $\{S(s) : s \geq 0\}$ on Φ is that

(1) $\mathcal{D}(A)$ is dense in Φ.

(2) There exists a sequence of τ-compatible norms $\{\|\cdot\|_n : n \geq 0\}$ on Φ and $n_0 \geq 0$ such that for each $n \geq n_0$ the following two conditions hold:

(a) A is closable in $\Phi_{|n|}$.

(b) The closure A_n of A in $\Phi_{|n|}$ is the infinitesimal generator of a C_0-semigroup $\{S^n(s) : s \geq 0\}$ on $\Phi_{|n|}$ such that for $s \geq 0$ $S^n(s)$ maps Φ into Φ and its restriction to Φ coincides with $S(s)$.

The following is a perturbation result for $(C_0,1)$-semigroups on Φ.

Proposition 1.1. Let A be the infinitesimal generator of a $(C_0,1)$-semigroup $\{S(s) : s \geq 0\}$ on Φ. Let B be a continuous linear operator on Φ such that there exists a sequence of τ-compatible norms $\{\|\cdot\|_n : n \geq 0\}$ on Φ and $n_0 \geq 0$ such that for $n \geq n_0$ B can be extended to a continuous linear operator on $\Phi_{|n|}$. Then $A+B$ is the infinitesimal generator of a $(C_0,1)$-semigroup $\{P(s) : s \geq 0\}$ on Φ satisfying the integral equation

$$P(s)\phi = S(s)\phi + \int_0^s S(s-r)BP(r)\phi\,dr \quad \phi \in \Phi.$$

Proof: Use Theorems 1.1, 1.2 and the classical perturbation theorem for semigroups in Banach spaces (Theorem 3.4.2 in [17]). $\square$

We now consider the construction of reversed evolution systems on Φ. In order to do this we need to introduce the following concept.

Definition 1.3. A family $\{A(t)\}_{t\geq 0}$ of infinitesimal generators of $(C_0,1)$-semigroups $\{S_t(s) : s\geq 0\}_{t\geq 0}$ on Φ is called *stable* if there exists a sequence of τ-compatible norms $\{\|\cdot\|_n : n\geq 0\}$ on Φ such that for each $T>0$ there exists $q_0\geq 0$ and for $q\geq q_0$ there are constants $M_q = M_q(T)\geq 1$ and $\sigma_q = \sigma_q(T)$ satisfying the following condition

$$(1.10) \qquad \Big\|\prod_{j=1}^{k} S_{t_j}(s_j)\phi\Big\|_q \leq M_q e^{\sigma_q \sum_{j=1}^{k} s_j}\|\phi\|_q \quad \text{for all } \phi\in\Phi,\ s_j\geq 0$$

whenever $0\leq t_1\leq t_2\leq\ldots\leq t_k\leq T$, $k\geq 0$. Here and in the sequel the time ordered product $\Pi_{j=1}^{k}S_{t_j}(s_j)\phi$ is $S_{t_1}(s_1)S_{t_2}(s_2)\ldots S_{t_k}(s_k)\phi$. The family $\{A(t)\}_{t\geq 0}$ is said to be *uniformly stable* if for each $q\geq 0$ M_q and σ_q are independent of T and (1.10) holds for $0\leq t_1\leq t_2\leq\ldots\leq t_k<\infty$. In either case we call M_q and σ_q $q\geq 0$ the *stability constants*.

Remark 1.1. In the literature on evolution systems in Banach spaces (see [16] or [17]) the product $\Pi_{j=1}^{k}B(t_j)$ for $0\leq t_1\leq\ldots\leq t_k$ is taken in descending order, i.e. $B(t_k)B(t_{k-1})\ldots B(t_1)$. Some results (as for example the analogous ones for Banach spaces of the next three propositions) remain true whatever the order in which the product is taken. However, in the construction of reversed evolution systems (Theorem 1.3 below) the order of this product is important and will be taken in "increasing order" as explained in Definition 1.3.

In the construction of evolution systems in a Banach space X the definition of stability of a family A(t) of infinitesimal generators of C_0-semigroups in X is given in terms of the resolvents $R(\lambda;A(t))$. However, in a nuclear Fréchet space Φ (or more generally in a locally convex space) the resolvent of an operator A(t) might not exist even when A(t) does generate a semigroup $S_t(s)$ on Φ. (See [1]). Nevertheless, an equivalent condition to (1.10) can be given in terms of the resolvents $R(\lambda;A_q(t))$ of the corresponding infinitesimal generators $A_q(t)$ on each of the Banach spaces $\Phi_{|q|}$.

Proposition 1.2. The condition (1.10) is equivalent to the following: $(\sigma_q,\infty) \subset \rho(A_q(t))$ for $0 \le t \le T$ and

$$(1.11) \qquad \left\| \prod_{j=1}^{k} R(\lambda;A_q(t_j))\phi \right\|_q \le M_q \|\phi\|_q (\lambda - \sigma_q)^{-k} \quad \lambda > \sigma_q, \ \phi \in \Phi$$

where $\{t_j\}$ are as in Definition 1.3 and $k \ge 0$.

The proof follows on the lines of Proposition 4.3.1 in Tanabe [17] for each $q \ge q_0$.

The following two criteria are useful in testing the stability of a family of operators $\{A(t)\}_{t\ge 0}$ on Φ.

Proposition 1.3. Let $\{A(t)\}_{t\ge 0}$ be a family of infinitesimal generators of $(C_0,1)$-semigroups $\{S_t(s) : s \ge 0\}_{t\ge 0}$ on Φ. Let $\{\|\cdot\|_n : n\ge 0\}$ be a sequence of τ-compatible norms on Φ such that for each $T > 0$ there exists $q_0 \ge 0$ and for $q \ge q_0$ there is a constant $\sigma_q = \sigma_q(T)$ satisfying the condition

$$(1.12) \qquad \|S_t(s)\phi\|_q \le e^{\sigma_q s} \|\phi\|_q \quad \phi \in \Phi, \ s \ge 0, \ 0 \le t \le T.$$

Then $\{A(t)\}_{t\geq 0}$ is a stable family on Φ. If moreover for each $q \geq 0$ σ_q is independent of T then $\{A(t)\}_{t\geq 0}$ is uniformly stable.

Proof: Using (1.12) in $\|\Pi_{j=1}^{k} S_{t_j}(s_j)\phi\|_q$ we obtain (1.10) with $M_q = 1$.

Proposition 1.4. Let $\{A(t)\}_{t\geq 0}$ be a family of infinitesimal generators of $(C_0,1)$-semigroups $\{S_t(s) : s \geq 0\}_{t\geq 0}$ on Φ stable with respect to the sequence of norms $\{\|\cdot\|_n : n \geq 0\}$. Let $\{B(t)\}_{t\geq 0}$ be a family of continuous linear operators on Φ. Assume there exists $q_0' \geq 0$ such that for $q \geq q_0'$ and $T > 0$ $\{B(t)\}_{0\leq t\leq T}$ can be extended to a family of uniformly bounded operators from $\Phi_{|q|}$ to $\Phi_{|q|}$. Then $\{A(t)+B(t)\}_{t\geq 0}$ is a stable family of infinitesimal generators of $(C_0,1)$-semigroups $\{V_t(s) : s \geq 0\}_{t\geq 0}$ on Φ.

Proof: Let $q \geq q_0'$ and also denote by $B(t)$ the extension of $B(t)$ from $\Phi_{|q|}$ to $\Phi_{|q|}$. Let

$$K_q(T) = \sup_{0\leq t\leq T} \|B(t)\|_{L(\Phi_{|q|},\Phi_{|q|})}.$$

From Proposition 1.1 for each $t \geq 0$ $A(t)+B(t)$ is the infinitesimal generator of a $(C_0,1)$-semigroup $\{P_t(s) : s \geq 0\}_{t\geq 0}$ on Φ. For each $T > 0$ let σ_q, M_q $q \geq q_0$ be the stability constants of the family $\{A(t)\}_{t\geq 0}$. We notice that Proposition 4.3.3 in [17] (see also Theorem 5.2.3 in [16]) remains true if the product of the corresponding operators is taken in increasing order. Then using this proposition we have that for each $q \geq \max(q_0, q_0')$

$$\|\prod_{j=1}^{k} P_{t_j}(s_j)\phi\|_q \leq M_q e^{(\sigma_q + K_q(T)M_q)\sum_{j=1}^{k} s_j} \|\phi\|_q \quad \phi \in \Phi,\ s_j \geq 0$$

$0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k \geq 1$. Hence the family $\{A(t)+B(t)\}_{t\geq 0}$ is stable with stability constants M_q, $\sigma_q + K_q(T)M_q$ $q \geq \max(q_0, q_0')$.

The main result of this section is the following theorem which gives sufficient conditions for the existence of reversed evolution systems on Φ. Its proof is on the lines of the construction of evolution systems on Banach spaces following an idea due to T. Kato (see [16] and [17]).

<u>Theorem 1.3</u>. Let $\{A(t)\}_{t\geq 0}$ be a family of continuous linear operators on Φ such that for each $t \geq 0$ $A(t)$ is the infinitesimal generator of a $(C_0,1)$-semigroup on Φ. Let $\{\|\cdot\|_n : n \geq 0\}$ be a sequence of τ-compatible norms on Φ such that the following two conditions hold:

(a) $\{A(t)\}_{t\geq 0}$ is a stable family with respect to $\{\|\cdot\|_n : n \geq 0\}$.

(b) For each $q \geq 0$ there exists $p \geq q$ such that for each $t \geq 0$ $A(t)$ has a *continuous* linear extension from $\Phi_{|p|}$ to $\Phi_{|q|}$ (also denoted by $A(t)$) and $t \to A(t)$ is $L(\Phi_{|p|}, \Phi_{|q|})$-continuous.

Then there exists a unique reversed evolution system $\{T(s,t) : 0 \leq s \leq t < \infty\}$ on Φ such that for each $T > 0$ the following three conditions are satisfied:

(1) For some $q_0 \geq 0$ and all $q \geq q_0$

$$\|T(s,t)\phi\|_q \leq M_q e^{\sigma_q(t-s)} \|\phi\|_q \quad \text{for all } \phi \in \Phi,\ 0 \leq s \leq t \leq T,$$

where $M_q = M_q(T)$ and $\sigma_q = \sigma_q(T)$ are the stability constants;

(2) $\frac{d}{dt}T(s,t)\phi = T(s,t)A(t)\phi$ for all $\phi \in \Phi$, $0 \leq s \leq t \leq T$;

(3) $\frac{d}{ds}T(s,t)\phi = -A(s)T(s,t)\phi$ for all $\phi \in \Phi$, $0 \leq s \leq t \leq T$;

If moreover $\{A(t)\}_{t\geq 0}$ is a uniformly stable family then conditions (1)-(3) hold for $0 \leq s \leq t < \infty$.

<u>Proof</u>: Let $T > 0$ be fixed but otherwise arbitrary and let $q \geq 0$. Then by condition (b) there exists $p \geq q$ such that

(1.13) $\quad \|A(t) - A(s)\|_{L(\Phi_{|p|}, \Phi_{|q|})} \xrightarrow[s\to t]{} 0$ uniformly in $t \in [0,T]$.

Let $t_k^n = \frac{k}{n}T$ $k = 0,1,\ldots, n$ and define the following step function approximation of $A(t)$:

$$A_n(t) = \begin{cases} A(t_k^n) & t_k^n \leq t < t_{k+1}^n \quad k = 0,1,\ldots, n-1 \\ A(t) & t = T. \end{cases}$$

Then by (1.13) for each $q \geq 0$ there exists $p \geq q$ such that

(1.14) $\quad \|A(t) - A_n(t)\|_{L(\Phi_{|p|}, \Phi_{|q|})} \xrightarrow[n\to\infty]{} 0$

uniformly in $t \in [0,T]$.

For each $t \geq 0$ let $\{S_t(s) : s \geq 0\}$ be the $(C_0,1)$-semigroup on Φ generated by $A(t)$. For $n \geq 1$ define the two parameter family of operators

$$T_n(s,t) = \begin{cases} S_{t_j^n}(t-s) & t_j^n \leq s \leq t \leq t_{j+1}^n \\ \\ S_{t_\ell^n}(t_{\ell+1}^n - s) \prod_{j=\ell+1}^{k-1} S_{t_j^n}(\frac{T}{n}) S_{t_k^n}(t - t_k^n) & \text{for } k > \ell \\ & t_k^n \leq t \leq t_{k+1}^n \\ & t_\ell^n \leq s \leq t_{\ell+1}^n \end{cases}$$

Using the semigroup property of $\{S_t(s) : s \geq 0\}$ it is easy to verify the reversed evolution property of $T_n(s,t)$ i.e.

$$T_n(s,t) = T_n(s,r)T_n(r,t) \quad 0 \leq s \leq r \leq t \leq T, \; T_n(t,t) = I,$$

and using the continuity of the map $s \to S_t(s)\phi$ for each $\phi \in \Phi$ and $t \geq 0$ it follows that for each $\phi \in \Phi$

(1.15) $\quad (s,t) \to T_n(s,t)\phi$ is continuous in Φ, $0 \leq s \leq t \leq T$.

Thus for each $n \geq 1$ $\{T_n(s,t) : 0 \leq s \leq t \leq T\}$ is a reversed evolution system on Φ. Moreover, from the definition of $T_n(s,t)$ and the fact that $A(t)$ is the infinitesimal generator of $\{S_t(s) : s \geq 0\}$ we obtain the following

(1.16) $$\frac{d}{dt}T_n(s,t)\phi = T_n(s,t)A_n(t)\phi \quad t \neq t_j^n \quad j = 1,\ldots, n, \; 0 \leq s \leq t \leq T, \; \phi \in \Phi$$

and

(1.17) $$\frac{d}{ds}T_n(s,t)\phi = -A_n(s)T_n(s,t)\phi \quad s \neq t_j^n \quad j = 1,\ldots, n, \; 0 \leq s \leq t \leq T, \; \phi \in \Phi.$$

Next for each $\phi \in \Phi$ the map $r \to T_n(s,r)T_m(r,t)\phi$ is differentiable except for a finite number of values of r and

$$\frac{d}{dr}(T_n(s,r)T_m(r,t)\phi) = T_n(s,r)(A_n(r) - A_m(r))T_m(r,t)\phi.$$

Then

$$T_n(s,t)\phi - T_m(s,t)\phi = \int_s^t T_n(s,r)(A_n(r) - A_m(r))T_m(r,t)\phi\, dr \quad \phi \in \Phi, \; 0 \leq s \leq t \leq T$$

and for each $q \geq 0$ and $\phi \in \Phi$

(1.18) $$\|T_n(s,t)\phi - T_m(s,t)\phi\|_q \leq \int_s^t \|T_n(s,r)(A_n(r) - A_m(r))T_m(r,t)\phi\|_q\, dr \quad 0 \leq s \leq t \leq T.$$

Now using the definition of $T_n(s,t)$ and the stability condition (1.10) we have that for each $q \geq q_0$ (where q_0 is given by the stability condition) and $0 \leq s \leq r \leq t \leq T$

$$(1.19) \quad \|T_n(s,r)(A_n(r) - A_m(r))T_m(r,t)\phi\|_q \leq M_q e^{\sigma_q(r-s)} \|(A_n(r) - A_m(r))T_m(r,t)\phi\|_q .$$

Using condition (b) we have that for each $q \geq q_0$ there exists $p \geq q$ such that for $0 \leq s \leq r \leq T$ and $\phi \in \Phi$

$$\|(A_n(r) - A_m(r))T_m(r,t)\phi\|_q \leq \|A_n(r) - A_m(r)\|_{L(\Phi_{|p|},\Phi_{|q|})} \|T_m(r,t)\phi\|_p$$

and using again the definition of $T_m(r,t)$ and the stability condition (1.10) we have that for $\phi \in \Phi$ and $0 \leq r \leq t \leq T$

$$(1.20) \quad \|T_m(r,t)\phi\|_p \leq M_p e^{\sigma_p(t-r)} \|\phi\|_p .$$

Then taking $M = \max(M_p, M_q)$ and $\sigma = \max(\sigma_p, \sigma_q)$ using the last three inequalities in (1.18) we obtain that for $0 \leq s \leq t \leq T$ and $\phi \in \Phi$

$$(1.21) \quad \|T_n(s,t)\phi - T_m(s,t)\phi\|_q \leq M^2 e^{\sigma(t-s)} \|\phi\|_p \int_s^t \|A_n(r) - A_m(r)\|_{L(\Phi_{|p|},\Phi_{|q|})} dr$$

which, using (1.10), goes to zero as n,m goes to infinite.

Hence for each $\phi \in \Phi$, $q \geq q_0$ and $0 \leq s \leq t \leq T$ $\{T_n(s,t)\phi\}_{n \geq 1} \subset \Phi$ is a Cauchy sequence in $\Phi_{|q|}$ and therefore a Cauchy sequence in Φ. Thus for each $\phi \in \Phi$ and $0 \leq s \leq t \leq T$ define the reversed evolution system

$$(1.22) \quad T(s,t)\phi = \lim_{n\to\infty} T_n(s,t)\phi \quad \text{(limit in } \Phi).$$

Then by definition we have that $T(s,t)\Phi \subset \Phi$ and using (1.20) we have (1) in the Theorem which also shows that $T(s,t) \in L(\Phi,\Phi)$. Properties (1.3) and (1.4) follow since $T_n(s,t)$ satisfies them.

Before proving (2)-(3) in the theorem we make the following observation: The system $\{T(s,t)\} = \{T^{T}(s,t)\}$ in (1.22) is defined for $0 \le s \le t \le T$ and would appear to depend on the interval $[0,T]$. We now show that under the stability condition on the family $\{A(t)\}_{t\ge 0}$,

(1.23) $\quad T^{T}(s,t)\phi = T^{T'}(s,t)\phi$ for all $\phi \in \Phi$, $0 \le s \le t \le T \le T'$.

Let $T' > T > 0$ and define $\tilde{t}_k^n = \frac{k}{n}T'$ $\quad k = 0,1,\ldots,n-1$

$$\tilde{A}_n(t) = A(\tilde{t}_k^n) \quad \tilde{t}_k^n \le t < \tilde{t}_{k+1}^n \quad k = 0,1,\ldots,n-1$$

$$\tilde{A}_n(T') = A(T')$$

$$\tilde{T}_n(s,t) = \begin{cases} S_{\tilde{t}_j^n}(t-s) & \tilde{t}_j^n \le s \le t \le \tilde{t}_{j+1}^n \\ S_{\tilde{t}_\ell^n}(\tilde{t}_{\ell+1}^n - s) \prod_{j=\ell+1}^{k-1} S_{\tilde{t}_k^n}\left(\frac{T'}{n}\right) S_{\tilde{t}_k^n}(t - \tilde{t}_k^n) & k > \ell \quad \tilde{t}_k^n \le t \le \tilde{t}_{k+1}^n \\ & \tilde{t}_\ell^n \le s \le \tilde{t}_{\ell+1}^n . \end{cases}$$

Using (1.13), for each $q \ge 0$ there exists $p \ge q$ such that

(1.24) $\quad \|\tilde{A}_n(t) - A_n(t)\|_{L(\Phi_{|p|},\Phi_{|q|})} \underset{n\to\infty}{\to} 0$

uniformly in $0 \le t \le T \le T'$. Next for each $\phi \in \Phi$ the map $r \to T_n(s,r)\tilde{T}_n(r,t)\phi$ is continuously differentiable except for a finite number of values of r and

$$\frac{d}{dr}(T_n(s,r)\tilde{T}_n(r,t)\phi) = T_n(s,r)(A_n(r) - \tilde{A}_n(r))\tilde{T}_n(r,t)\phi$$

i.e.,

$$T_n(s,t)\phi - \tilde{T}_n(s,t)\phi = \int_s^t T_n(s,r)(A_n(r) - \tilde{A}_n(r))\tilde{T}_n(r,t)\phi \, dr \quad \text{for all } \phi \in \Phi,$$
$$0 \le s \le t \le T \le T'.$$

Then using (b) and the stability condition, for each $q \ge 0$ there exists $p \ge q$ such that

$$\|T_n(s,t)\phi - \tilde{T}_n(s,t)\phi\|_q \le M_q(T)M_p(T')e^{\sigma(t-s)}\|\phi\|_p \int_s^t \|A_n(r) - \tilde{A}_n(r)\|_{L(\Phi_{|p|},\Phi_{|q|})} dr$$

where $\sigma = \max(\sigma_q(T), \sigma_p(T'))$ and $M_q(T)$, $M_p(T')$, $\sigma_q(T)$ and $\sigma_p(T')$ are the stability constants. Then using (1.24) and (1.22) we obtain (1.23).

Now we return to the proof of the theorem. To prove (2) let $q \ge 0$ and $\phi \in \Phi$, then since for each $t \ge 0$ $A(t)$ is the infinitesimal generator of $\{S_t(s) : s \ge 0\}$, the function $r \to S_s(r-s)T_n(r,t)\phi$ is differentiable except for a finite number of values of r and we have that

$$\|T_n(t,s)\phi - S_s(t-s)\phi\|_q = \left\|\int_s^t \frac{d}{dr}\{S_s(r-s)T_n(r,t)\phi\}dr\right\|_q$$

$$= \left\|\int_s^t S_s(r-s)(A(s) - A_n(r))T_n(r,t)\phi dr\right\|_q$$

$$\le M^2 e^{\sigma(t-s)} \|\phi\|_p \int_s^t \|A_n(r) - A(s)\|_{L(\Phi_{|p|},\Phi_{|q|})} dr$$

for all $\phi \in \Phi$, $0 \le s \le t \le T$ where M, σ and p are as in (1.21). Then using (1.14) and (1.22) we obtain that for $0 \le s \le t \le T$ and $\phi \in \Phi$

$$\text{(1.25)} \quad \|T(s,t)\phi - S_s(t-s)\phi\|_q \le M^2 e^{\sigma(t-s)}\|\phi\|_p \int_s^t \|A(r) - A(s)\|_{L(\Phi_{|p|},\Phi_{|q|})} dr.$$

Hence by dividing both sides of (1.25) by $t-s$ and letting $t \to s$ we have that for each $q \ge 0$ and $\phi \in \Phi$

$$\frac{1}{t-s}\|T(s,t)\phi - S_s(t-s)\phi\|_q \to 0 \quad \text{as } t \downarrow s.$$

Then $\frac{d^+}{dt}T(s,t)\phi$ exists in Φ and since $\frac{d}{dt}S_s(t-s)\phi = A(s)S_s(t-s)\phi$ we have that

(1.26) $$\frac{d^+}{dt}T(s,t)\phi\Big|_{t=s} = A(s)\phi \quad \text{(limit in } \Phi) \text{ for all } \phi \in \Phi.$$

In a similar way one shows that for each $\phi \in \Phi$

(1.27) $$\frac{d^-}{ds}T(s,t)\phi\Big|_{s=t} = -A(t)\phi.$$

Next using (1.26) we have that for $\phi \in \Phi$ and $s \le t$

(1.28) $$\frac{d^+}{dt}T(s,t)\phi = \lim_{h\downarrow 0}\frac{1}{h}\{T(s,t+h)\phi - T(s,t)\phi\}$$

$$= T(s,t)\lim_{h\downarrow 0}\frac{1}{h}\{T(t,t+h)\phi - \phi\} = T(s,t)A(t)\phi.$$

Now for $s < t$, using (1) in the Theorem, (1.4) and (1.27) we have that for each $q \ge q_0$ and $\phi \in \Phi$

$$\limsup_{h\uparrow 0}\left\|T(s,t+h)\left\{\frac{\phi - T(t+h,t)\phi}{h}\right\} - T(s,t)A(t)\phi\right\|_q$$

$$= \limsup_{h\uparrow 0}\left\|T(s,t+h)\left\{\frac{\phi - T(t+h,t)\phi}{h}\right\} - T(s,t+h)T(t+h,t)A(t)\phi\right\|_q$$

$$\le \limsup_{h\uparrow 0} M_q e^{\sigma_q(t+h-s)}\left\|\frac{\phi - T(t+h,t)\phi}{h} - T(t+h,t)A(t)\phi\right\|_q = 0.$$

Then for $s < t$ and $\phi \in \Phi$

$$\frac{d^-}{dt}T(s,t)\phi = \lim_{h\uparrow 0}\frac{1}{h}\{T(s,t+h)\phi - T(s,t)\phi\} = T(s,t)A(t)\phi$$

which together with (1.28) imply (2) in the Theorem. In a similar way (3) is proved.

To prove uniqueness of the system $\{T(s,t) : 0 \le s \le t \le T\}$, suppose that $V(s,t)$ is a reversed evolution system with generators $\{A(t)\}_{t\ge 0}$ satisfying (1) and the same forward and backward equations (2) and (3). Then except for a finite number of values of r the map $r \to V(s,r)T_n(r,t)\phi$ is continuously differentiable for each $\phi \in \Phi$ and we obtain

$$V(s,t)\phi - T_n(s,t)\phi = \int_s^t V(s,r)(A_n(r) - A(r))T_n(r,t)\phi dr \quad \text{for all } \phi \in \Phi,\ 0 \le s \le t \le T.$$

Using (1.20) for $T_n(s,t)$ and (1) in the theorem for $V(s,t)$ we obtain that for each $q \ge 0$ there exists $p \ge q$ such that

$$\| V(s,t)\phi - T_n(s,t)\phi \|_q \le M_q(T)M_p(T)e^{\sigma(t-s)} \|\phi\|_p \int_s^t \|A(r) - A_n(r)\|_{L(\Phi_{|p|},\Phi_{|q|})} dr.$$

Then by (1.14) and (1.22) we have

$$V(s,t)\phi = T(s,t)\phi \text{ for all } \phi \in \Phi,\ 0 \le s \le t \le T,\quad T > 0. \qquad \square$$

<u>Definition 1.4</u>. A reversed evolution system $\{T(s,t) : 0 \le s \le t < \infty\}$ on Φ satisfying (1)-(3) in Theorem 1.3 is called a $(C_0,1)$-*reversed evolution system*.

The following is a perturbation result for $(C_0,1)$-reversed evolution systems on Φ.

<u>Theorem 1.4</u>. Let $\{A(t)\}_{t\ge 0}$ be a family of continuous linear operators on Φ satisfying the conditions of Theorem 1.3. Let $\{B(t)\}_{t\ge 0}$ be a family of continuous linear operators on Φ such that there exists $q_0 \ge 0$ and for $q \ge q_0$ and $t \ge 0$ $B(t)$ has a continuous linear extension to $\Phi_{|q|}$ and the map $t \to B(t)$ is $L(\Phi_{|q|},\Phi_{|q|})$-continuous. Then there exists a unique $(C_0,1)$-reversed evolution system $V(s,t)$ on Φ satisfying (1)-(3) in Theorem 1.3 for the stable family $\{A(t) + B(t)\}_{t\ge 0}$ of infinitesimal generators of $(C_0,1)$-semi-

groups. Moreover $V(s,t)$ satisfies the integral equation

$$(1.29) \qquad V(s,t)\phi = T(s,t)\phi + \int_s^t T(s,r)B(r)V(r,t)\phi\,dr \quad \text{for all } \phi \in \Phi,\ 0 \le s \le t$$

where $T(s,t)$ is the $(C_0,1)$-reversed evolution system generated by the family $\{A(t)\}_{t\ge 0}$.

<u>Proof</u>: Under the conditions on $\{B(t)\}_{t\ge 0}$ and using Proposition 1.4, the family $\{A(t)+B(t)\}_{t\ge 0}$ satisfies the conditions of Theorem 1.3 which proves the existence of the reversed evolution system $V(s,t)$ satisfying properties (1)-(3) in Theorem 1.3 for the family $\{A(t)+B(t)\}_{t\ge 0}$.

To show that V satisfies (1.29), for each $T > 0$ and $\phi \in \Phi$ define for $0 \le s \le t \le T$

$$V_0(s,t)\phi = T(s,t)\phi$$

$$(1.30) \qquad V^{(m)}(0,t)\phi = \int_s^t T(s,r)B(r)V^{(m-1)}(r,t)\phi\,dr \qquad m \ge 1$$

$$\tilde{V}(s,t)\phi = \sum_{m=0}^{\infty} V^{(m)}(s,t)\phi \quad \text{(convergence in } \Phi\text{)}.$$

Applying (1) in Theorem 1.3 to $T(s,t)$ and using the continuity of the map $t \to B(t)$ in $L(\Phi_{|q|},\Phi_{|q|})$ for $q \ge q_0$, we have that for $q \ge q_0$, $0 \le s \le t \le T$ and $m \ge 1$

$$\|V^{(m)}(s,t)\phi\|_q \le M_q e^{\sigma_q(t-s)}(K_q(T)M_q)^m \|\phi\| \frac{(t-s)^m}{m!} \quad \text{for all } \phi \in \Phi$$

where

$$K_q(T) = \sup_{0\le t\le T} \|B(t)\|_{L(\Phi_{|q|},\Phi_{|q|})}.$$

Then for each $\phi \in \Phi$ the series (1.30) converges on Φ uniformly in $0 \le s \le t \le T$ and therefore $\tilde{V}$ satisfies (1.29). Moreover for $q \ge q_0$

$$\|\tilde{V}(s,t)\phi\|_q \le M_q e^{(\sigma_q + K_q(T)M_q)(t-s)} \|\phi\|_q \quad \text{for all } \phi \in \Phi,\ 0 \le s \le t \le T$$

which shows (1) for $\tilde{V}$ in Theorem 1.3.

It is not difficult to prove that $\tilde{V}$ also satisfies (2) and (3) in Theorem 1.3 which shows that $\tilde{V} = V$. $\square$

As a consequence of Theorem 1.4 we now obtain the reversed evolution system generated by a family of operators of the form $\{A + B(t)\}_{t\ge 0}$. Following the terminology for Banach spaces used by Curtain and Pritchard [3] we call these operators "quasi-generators".

<u>Corollary 1.1</u>. Let A be a continuous linear operator on Φ which is the infinitesimal generator of a $(C_0,1)$ semigroup $\{S(s) : s \ge 0\}$ on Φ. Let $\{B(t)\}_{t\ge 0}$ be a family of continuous linear operators on Φ such that there exists a sequence of τ-compatible norms $\{\|\cdot\|_n : n \ge 0\}$ on Φ and $q_0 \ge 0$ such that for $q \ge q_0$ and $t \ge 0$ $B(t)$ has a continuous linear extension to $\Phi_{|q|}$ and the map $t \to B(t)$ is $L(\Phi_{|q|}, \Phi_{|q|})$ continuous. Then the family $\{A + B(t)\}_{t\ge 0}$ is stable and there exists a unique $(C_0,1)$-reversed evolution system $T(s,t)$ on Φ satisfying (1)-(3) in Theorem 1.3 for the family $\{A + B(t)\}_{t\ge 0}$. Moreover $T(s,t)$ satisfies the integral equation

$$(1.31) \qquad T(s,t)\phi = S(t-s)\phi + \int_s^t S(r-s)B(r)T(r,t)\phi\,dr \quad \text{for all } \phi \in \Phi,\ 0 \le s \le t.$$

The following result will be used in proving uniqueness of the solution for the stochastic evolution equation in the next section.

<u>Proposition 1.5</u>. Let $\{A(t)\}_{t\geq 0}$ be a family of linear operators as in Theorem 1.3. Then for any $X_0 \in \Phi'$ the Φ'-valued initial value problem

$$X_t = X_0 + \int_0^t A_s' X_s ds$$

i.e.,

$$X_t[\phi] = X_0[\phi] + \int_0^t X_s[A_s\phi]ds \quad \text{for all } \phi \in \Phi$$

has a unique Φ'-valued solution given by $\xi_t = T'(t,0)X_0$ where $\{T(s,t) : 0 \leq s \leq t < \infty\}$ is the $(C_0,1)$-reversed evolution system on Φ generated by the family $\{A(t)\}_{t\geq 0}$.

The proof of the above proposition follows easily from the following lemma.

<u>Lemma 1.1</u>. Let $\{A(t)\}_{t\geq 0}$ be a family of continuous linear operators on Φ satisfying the conditions of Theorem 1.3 and let $\{T(s,t) : 0 \leq s \leq t < \infty\}$ be the $(C_0,1)$-reversed evolution system generated by it. Let B be any continuous linear operator from Φ to Φ. Then for each $F \in \Phi'$ and $0 \leq u \leq t$ the following identities hold:

(a) $$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BT(u,s)A(s)\phi]ds \quad \text{for all } \phi \in \Phi$$

(b) $$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BA(s)T(s,t)\phi]ds \quad \text{for all } \phi \in \Phi.$$

<u>Proof</u>: Use the forward and backward equations given by (2) and (3) in Theorem 1.3.

2. STOCHASTIC EVOLUTION EQUATIONS

Let (Ω, F, P) be a complete probability space with a right continuous filtration $(F_t)_{t\geq 0}$, F_0 containing all the P-null sets of F. Let $(\Phi, |\cdot|_n\ n\geq 0)$ be a countably Hilbertian nuclear space and Φ_n, Φ_n' $n\geq 0$ and Φ' be as in Section 1. Let $M = (M_t)_{t\geq 0}$ be a Φ'-valued martingale with respect to F_t, i.e. for each $\phi\in\Phi$ $(M_t[\phi], F_t)_{t\geq 0}$ is a real valued martingale. This section concerns the solution of the Φ'-valued stochastic evolution equation

$$(2.1)\qquad \begin{aligned} d\xi_t &= A'(t)\xi_t dt + dM_t \quad t>0 \\ \xi_0 &= \gamma \end{aligned}$$

where γ is a Φ'-valued random variable, $\{A(t)\}_{t\geq 0}$ is a family of continuous linear operators on Φ generating a $(C_0,1)$-reversed evolution system $\{T(s,t) : 0\leq s\leq t<\infty\}$ on Φ and $\{A'(t)\}_{t\geq 0}$ are defined by the relation (1.2). We also consider perturbations of (2.1) i.e.

$$(2.2)\qquad d\xi_t = A'(t)\xi_t dt + B'(t)\xi_t dt + dM_t$$

where $\{B(t)\}_{t\geq 0}$ is a family of continuous linear operators on Φ. Our results also include the case when $A(t) = A$ $t\geq 0$ and A is the infinitesimal generator of a $(C_0,1)$-semigroup on Φ (Corollary 2.2).

To begin with the study of stochastic evolution equations driven by Φ'-valued martingales we first recall from Mitoma [12] some properties of such Φ'-valued processes. We will denote by $D(T;\Psi')$ (respectively $C(T;\Psi')$) the space of right continuous processes with left hand limits (respectively, continuous processes) indexed by $T([0,T]$ or $[0,\infty))$ and with values in Ψ' (Φ' or Φ_q').

Proposition 2.1. (a) If $M = (M_t)_{t\geq 0}$ is a Φ'-valued martingale there exists a Φ'-valued version, also denoted by M, such that the following two conditions hold:

(i) For each $T > 0$ there exists $m_T > 0$ and $M^T_{\bullet} \in D([0,T];\Phi'_{m_T})$ a.s. where $M^T_{\bullet} = (M_t : 0 \leq t \leq T)$.

(ii) $M_{\bullet} \in D([0,\infty);\Phi')$ a.s.

(b) If moreover $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$, $t \geq 0$ then for each $T > 0$ there exists $q_T > 0$ such that $M^T_{\bullet} \in D([0,T];\Phi'_{q_T})$ a.s. and

$$E(\sup_{0\leq t\leq T} |M_t|^2_{-q_T}) < \infty.$$

In view of the above proposition, from now on we shall always consider Φ'-valued martingales in $D([0,\infty);\Phi')$.

We now give the meaning of the solution of the stochastic evolution equation (2.1). A similar definition is given for the solution of the perturbed equation (2.2).

Definition 2.1. We say that the stochastic evolution equation (2.1) has a Φ'-*valued solution* $\xi = (\xi_t)_{t\geq 0}$ if ξ satisfies the following conditions:

(a) ξ_t is Φ'-valued, progressively measurable and F_t-adapted.

(b) $\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A(s)\phi]ds + M_t[\phi]$ for all $\phi \in \Phi$, $t \geq 0$ a.s.

In Theorem 2.1 below we will prove that the unique solution of (2.1) is given by the so called "evolution or mild solution"

$$\xi_t = T'(t,0)\gamma + \int_0^t T'(t,s)dM_s$$

where $\{T'(t,s) : 0 \leq s \leq t < \infty\}$ is the evolution system on Φ' adjoint to the $(C_0,1)$-reversed evolution system $\{T(s,t) : 0 \leq s \leq t < \infty\}$ on Φ, and M is a Φ'-valued martingale such that $E(M_t[\phi])^2 < \infty$ for all $\phi \in \Phi$,

$t \geq 0$. The above Φ'-valued integral is defined for each $T > 0$ and $0 \leq t \leq T$ as the sum of the L^2-convergent series

$$(2.3) \qquad \left(\int_0^t T'(t,s)dM_s\right)[\phi] = \int_0^t dM_s[T(s,t)\phi] = \sum_{j=1}^{\infty} \int_0^t \langle T(s,t)\phi, \phi_j\rangle_{p_T} dM_s[\phi_j] \quad \phi \in \Phi$$

where $p_T > q_T$ is such that the injection $\Phi_{p_T} \hookrightarrow \Phi_{q_T}$ is nuclear, q_T is as in Proposition 2.1(b) and $\{\phi_j\}_{j\geq 1} \subset \Phi$ is a CONS in Φ_{p_T}. The evolution solution has the important property of being a Φ'-valued semimartingale, i.e. for each $\phi \in \Phi$, $\xi_t[\phi]$ is a real valued semimartingale.

We now present the main result of this section concerning the solution of the stochastic evolution equation (2.1).

<u>Theorem 2.1</u>. Assume the following conditions:

(A1) γ is a Φ'-valued F_0-measurable random element such that for some $r_0 > 0$ $E|\gamma|^2_{-r_0} < \infty$.

(A2) $M = (M_t)_{t\geq 0}$ is a Φ'-valued martingale such that $M_0 = 0$ and for each $t \geq 0$ and $\phi \in \Phi$, $E(M_t[\phi])^2 < \infty$.

(A3) $\{A(t)\}_{t\geq 0}$ is a family of continuous linear operators on Φ satisfying the following two conditions:

(a) $\{A(t)\}_{t\geq 0}$ is stable on Φ.

(b) For each $n \geq 0$ there exists $m \geq n$ such that for each $t \geq 0$ $A(t)$ has a continuous linear extension from Φ_m to Φ_n and the map $t \to A(t)$ is $L(\Phi_m, \Phi_n)$ continuous.

Then the stochastic evolution equation (2.1) has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ given by the evolution solution

$$(2.4) \qquad \xi_t = T'(t,0)\gamma + \int_0^t T'(t,s)dM_s$$

where $\{T(s,t) : 0 \le s \le t < \infty\}$ is the unique $(C_0,1)$-reversed evolution system generated by $\{A(t)\}_{t\ge 0}$ and given by Theorem 1.3. Moreover ξ has the following properties:

(1) $\xi_{\cdot} \in D([0,\infty):\Phi')$ a.s. and for each $T>0$ there exists $p_T > 0$ such that $\xi_{\cdot}^T \in D([0,T]:\Phi'_{p_T})$ a.s. and

$$E\left(\sup_{0\le t\le T} |\xi_t|^2_{-p_T}\right) < \infty.$$

(2) ξ given by (2.4) satisfies

$$\xi_t = M_t + \{T'(t,0)\gamma + \int_0^t T'(t,s)A'(s)M_s ds\}$$

i.e.,

$$\xi_t[\phi] = M_t[\phi] + \{\gamma[T(0,t)\phi] + \int_0^t M_s[A(s)T(s,t)\phi]ds\} \text{ for all } \phi\in\Phi.$$

Proof: By condition A3 and Theorem 1.3, the family $\{A(t)\}_{t\ge 0}$ generates a unique $(C_0,1)$-reversed evolution system $\{T(s,t): 0\le s\le t<\infty\}$ on Φ.

For $T>0$ let q_T given by Proposition 2.1(b) (we take $q_T \ge r_0$) and define

$$\Omega_1^T = \{\omega\in\Omega : M_{\cdot}^T \in D([0,T];\Phi'_{q_T})\}\cap\{\omega\in\Omega : |\gamma(\omega)|_{-r_0} < \infty\},$$

$$\text{(2.5)} \qquad C_T(\omega) = \sup_{0\le t\le T} |M_t(\omega)|_{-q_T}.$$

Then by A1 and Proposition 2.1(b) $P(\Omega_1^T) = 1$ and $C_T(\omega) < \infty$ for $\omega\in\Omega_1^T$. Moreover, using (1) in Theorem 1.3 and A3(b) there exists $r_T > q_T$ such that

$$\text{(2.6)} \qquad |A(s)T(s,t)\phi|_{q_T} \le M_{q_T} e^{\sigma_{q_T} T} K_{q_T} |\phi|_{r_T} \quad \text{for all } \phi\in\Phi,\ 0\le s\le t\le T$$

where

$$K_{q_T} = \sup_{0 \le t \le T} \|A(t)\|_{L(\Phi_{r_T}, \Phi_{q_T})} < \infty.$$

It is important to observe that if $q_T = q$ independent of T, then $r_T = r$ also is independent of T and (2.6) holds, although the constants M_{q_T}, σ_{q_T}, and K_{q_T} might still depend on T.

The proof of the theorem is completed in several steps.

Step 1 We first show that for each $t > 0$ the map

$$\phi \to \int_0^t M_s[A(s)T(s,t)\phi]ds$$

is continuous and linear on Φ a.s.: Let $T > 0$ be fixed but otherwise arbitrary and $\omega \in \Omega_1^T$. Using (2.5) and (2.6) we have that for $0 \le t \le T$

$$(2.7) \qquad \left|\int_0^t M_s(\omega)[A(s)T(s,t)\phi]ds\right| \le C_T(\omega) T N_1(T) |\phi|_{r_T} \quad \text{for all } \phi \in \Phi$$

where $N_1(T) = M_{q_T} e^{\sigma_{q_T} T} K_{q_T}$. Then for $\omega \in \Omega_1^T$ and $0 \le t \le T$

$$(2.8) \qquad Y_t(\omega)[\phi] := \int_0^t M_s(\omega)[A(s)T(s,t)\phi]ds \quad \phi \in \Phi$$

defines a continuous linear functional on Φ, i.e. $Y_t(\omega) \in \Phi'$. Moreover from (2.7) we have that for $\omega \in \Omega_1^T$

$$(2.9) \qquad \sup_{0 \le t \le T} |Y_t(\omega)[\phi]|^2 \le C_T^2(\omega) T^2 N_1^2(T) |\phi|_{r_T}^2 \quad \text{for all } \phi \in \Phi.$$

Since $T > 0$ is arbitrary then for each $t \ge 0$ $Y_t \in \Phi'$ a.s.

Step 2 For each $T > 0$ there exists $p_T > r_T$ such that

$$Y_\cdot^T := (Y_t : 0 \le t \le T) \in C([0,T] : \Phi'_{p_T}) \text{ a.s.}$$

Let $T > 0$, $\omega \in \Omega_1^T$ (we will sometimes suppress ω in the writing) and $t_0, t \in [0,T]$. Using Lemma 1.1(a) it is not difficult to show that (assuming $t_0 < t$)

$$Y_t[\phi] - Y_{t_0}[\phi] = \int_0^{t_0}\int_{t_0}^{t} M_u[A(u)T(u,s)A(s)\phi]dsdu + \int_{t_0}^{t} M_u[A(u)T(u,t)\phi]du \quad \text{for all } \phi \in \Phi.$$

Next using (2.7) and condition A3(b), there exists $\ell_T > r_T$ such that for some constant $N_2(T)$

(2.10) $\quad |Y_t(\omega)[\phi] - Y_{t_0}(\omega)[\phi]| \le C_T(\omega)N_2(T)T|t-t_0||\phi|_{\ell_T}$ for all $\phi \in \Phi$.

Then $Y_t(\omega)[\phi]$ is continuous in $0 \le t \le T$ for all $\phi \in \Phi$, $\omega \in \Omega_1^T$.

Let $p_T > r_T$ be such that the injection $\Phi_{p_T} \hookrightarrow \Phi_{r_T}$ is a Hilbert-Schmidt operator and let $\{\phi_j\}_{j\ge1} \subset \Phi$ be a CONS for Φ_{p_T}. Then from (2.9) for $\omega \in \Omega_1^T$

(2.11) $\quad \sum_{j=1}^{\infty} \sup_{0\le t\le T} (Y_t(\omega)[\phi_j])^2 \le C_T^2(\omega)T^2N_1^2(T)\sum_{j=1}^{\infty}|\phi_j|_{r_T}^2 < \infty.$

Then using the continuity of $Y_t(\omega)[\phi]$, (2.11) and the dominated convergence theorem we have that for $\omega \in \Omega_1^T$ and $t, t_0 \in [0,T]$

$$\lim_{t\to t_0}|Y_t(\omega) - Y_{t_0}(\omega)|_{-p_T}^2 = \lim_{t\to t_0}\sum_{j=1}^{\infty}(Y_t(\omega)[\phi_j] - Y_{t_0}(\omega)[\phi_j])^2 = 0.$$

Then we have shown that for each $T > 0$ there exists $p_T > r_T$ such that $Y_\cdot^T(\omega) \in C([0,T];\Phi'_{p_T})$ for $\omega \in \Omega_1^T$, $P(\Omega_1^T) = 1$.

<u>Step 3</u> We shall prove that there exists a Φ'-valued process ξ_t satisfying (b) in Definition 2.1.

Let $T_n \uparrow \infty$ and $\Omega_1 = \bigcap_{n=1}^{\infty}\Omega_1^{T_n}$, then $P(\Omega_1) = 1$, and from Step 2 we have that

$$Y_{\bullet}(\omega) \in C([0,\infty) : \Phi') \quad \omega \in \Omega_1.$$

Let $\Omega_2 = \{\omega \in \Omega : M_{\bullet}(\omega) \in D([0,\infty) : \Phi')\} \cap \{\omega \in \Omega : |\gamma(\omega)|_{-r_0} < \infty\}$. Then by (A1), (A2) and Proposition 2.1a(ii) $P(\Omega_2) = 1$. Let $\Omega^* = \Omega_1 \cap \Omega_2$, then $P(\Omega^*) = 1$ and for $\omega \in \Omega^*$

$$(2.12) \qquad \xi_t(\omega) = T'(t,0)\gamma(\omega) + Y_t(\omega) + M_t(\omega)$$

is a well defined element on Φ' for each $t \geq 0$. Since $Y_{\bullet} \in C([0,\infty) : \Phi')$ a.s. and $M_{\bullet} \in D([0,\infty) : \Phi')$ a.s. then $\xi_{\bullet} \in D([0,\infty) : \Phi')$ a.s. and

$$(2.13) \qquad \xi_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t M_s[A(s)T(s,t)\phi]ds + M_t[\phi] \text{ for all } \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

Next let $\omega \in \Omega^*$, $t > 0$ and $\phi \in \Phi$ (we will suppress ω in the writing). Applying Lemma 1.1(a) to $B = I$, $F = \gamma$ and $u = 0$ we have

$$(2.14) \qquad \gamma[T(0,t)\phi] = \gamma[\phi] + \int_0^t \gamma[T(0,s)A(s)\phi]ds$$

and taking $F = M_u$, $B = I$ in Lemma 1.1(a) we obtain

$$(2.15) \qquad M_u[A(u)T(u,t)\phi] = M_u[A(u)\phi] + \int_u^t M_u[A(u)T(u,s)A(s)\phi]ds.$$

Using (2.14) and (2.15) in (2.13) and applying Fubini's theorem we obtain

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \gamma[T(0,t)A(s)\phi]ds + \int_0^t \{M_s[A(s)\phi] + \int_0^s M_u[A(u)T(u,s)A(s)\phi]du\}ds + M_t[\phi]$$

$$= \gamma[\phi] + \int_0^t \xi_s[A(s)\phi]ds + M_t[\phi].$$

Then the process ξ_t given by (2.12) satisfies (b) in Definition 2.1.

Observe that the map $(t,\omega) \to \xi_t(\omega)$ is $\mathcal{B}(\Phi')/\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable and for each t, ξ_t is $\mathcal{F}_t^{M,\gamma}$-measurable where

$$F_t^{M,\gamma} = \sigma\{\gamma[\phi], M_s[\phi] : 0 \le s \le t,\ \phi \in \Phi\}.$$

Moreover, (2.12), Proposition 2.1(b) and Step 3 give that for each $T > 0$ there exists $p_T > 0$ such that

$$\xi_\cdot^T \in D([0,T]; \Phi'_{p_T}) \text{ a.s.}$$

Step 4 It remains to show that ξ_t defined by (2.12) and satisfying equation (b) of Definition (2.1), is given by the expression (2.4) so that the latter is the required solution. Uniqueness follows easily from Proposition 1.5. For each $T > 0$ let $p_T > q_T$ be such that the injection $\Phi_{p_T} \hookrightarrow \Phi_{q_T}$ is a nuclear operator. By (A2), (2.5) and Proposition 2.1(b)

(2.16) $$E(\sup_{0 \le t \le T} (M_t[\phi])^2) \le E(C_T)^2 |\phi|_{q_T}^2 < \infty \text{ for all } \phi \in \Phi.$$

First observe that the series

(2.17) $$\sum_{j=1}^{\infty} \int_0^t \langle T(s,t)\phi, \phi_j \rangle_{p_T} dM_s[\phi_j]$$

converges in $L^2(\Omega)$ for $\phi \in \Phi$. For, using (2.16), conclusion (1) of Theorem 1.3 and the nuclear property, we have that for $0 \le t \le T$ and $\phi \in \Phi$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left| E \int_0^t \langle T(s,t)\phi, \phi_j \rangle_{p_T} dM_s[\phi_j] \int_0^t \langle T(s,t)\phi, \phi_k \rangle_{p_T} dM_s[\phi_k] \right|$$

$$\le \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \{ E \int_0^t \langle T(s,t)\phi, \phi_j \rangle_{p_T}^2 d\langle M[\phi_j] \rangle_s \cdot E \int_0^t \langle T(s,t)\phi, \phi_k \rangle_{p_T}^2 d\langle M[\phi_k] \rangle_s \}^{\frac{1}{2}}$$

$$\le M_{p_T}^2 \|\phi\|_{p_T}^2 e^{2\sigma_p T} \{ E\langle M[\phi_j] \rangle_T\, E\langle M[\phi_k] \rangle_T \}^{\frac{1}{2}}$$

$$= M_{p_T}^2 \|\phi\|_{p_T}^2 e^{2\sigma_p T} \{ EM_T[\phi_j]^2 \cdot EM_T[\phi_k]^2 \}^{\frac{1}{2}}$$

$$\le E(C_T^2) M_{p_T}^2 e^{2\sigma_{p_T} T} \|\phi\|_{p_T}^2 \left(\sum_{j=1}^{\infty} \|\phi_j\|_{r_T}^2 \right)^2 < \infty.$$

Next using the backward equation (1.6) and Ito's formula, for each $\phi \in \Phi$ and $j \geq 1$ we have

$$(2.18) \quad \int_0^t \langle T(s,t)\phi,\phi_j\rangle_{p_T} dM_s[\phi_j] = M_t[\langle\phi,\phi_j\rangle_{p_T}\phi_j] - \int_0^t M_s[\phi_j]\frac{d}{ds}\langle T(s,t)\phi,\phi_j\rangle_{p_T} ds$$

$$= M_t[\langle\phi,\phi_j\rangle_{p_T}\phi_j] + \int_0^t M_s[\langle A(s)T(s,t)\phi,\phi_j\rangle_{p_T}\phi_j]ds.$$

But

$$(2.19) \quad E(M_t[\phi] - \sum_{j=1}^n M_t[\langle\phi,\phi_j\rangle_{p_T}\phi_j])^2 \leq E(C_T^2)\|\phi - \sum_{j=1}^n \langle\phi,\phi_j\rangle_{p_T}\phi_j\|_{q_T}^2$$

$$\leq E(C_T^2)\|\phi - \sum_{j=1}^n \langle\phi,\phi_j\rangle_{p_T}\phi_j\|_{p_T}^2 \xrightarrow[n\to\infty]{} 0.$$

Then from (2.17), (2.18) and (2.19) the series

$$\sum_{j=1}^\infty \int_0^t M_s[\langle A(s)T(s,t)\phi,\phi_j\rangle_{p_T}\phi_j]ds$$

converges also in $L^2(\Omega)$ and therefore, for each $\phi \in \Phi$

$$(2.20) \quad \sum_{j=1}^\infty \int_0^t \langle T(s,t)\phi,\phi_j\rangle_{p_T} dM_s[\phi_j] = M_t[\phi] + \int_0^t M_s[A(s)T(s,t)\phi]ds \quad \text{a.s.}$$

The assertion now follows from (2.13), (2.20) and (2.3).

<u>Remark 2.1.</u>

a).-From (2.10) we have that for $\omega \in \Omega_1^T$ $Y_t(\omega)[\phi]$ is a continuous real valued process of finite variation. It is not difficult to prove that $\gamma[T(0,t)\phi]$ is also a continuous process of finite variation on each finite interval. Then since they are F_t-adapted they are predictable and from (2.5) $\xi_t[\phi]$ is a real valued special semi-martingale with decomposition

$$\xi_t[\phi] = M_t[\phi] + V_t[\phi]$$

where $V_t[\phi] = \gamma[T(0,t)\phi] + Y_t[\phi]$.

b).-If the Φ'-valued martingale M is continuous then (i) in Theorem 2.1 holds if we replace the spaces $D([0,\cdot);\Phi')$ and $D([0,T];\Phi'_{p_T})$ by $D([0,\infty):\Phi')$ and $C([0,T]:\Phi'_{p_T})$ respectively.

We now obtain some easy but important consequences of the above Theorem.

<u>Theorem 2.2</u>. Assume (A1)-(A3) of Theorem 2.1 and that there exists $q \geq 0$ such that $M \in D([0,\infty):\Phi'_q)$ a.s. (or $C([0,\infty):\Phi'_q)$ a.s.). Then there exists $p > q$ such that the solution $\xi = (\xi_t)_{t \geq 0}$ of (2.1) satisfies the property $\xi_\cdot \in D([0,\infty):\Phi'_p)$ a.s. ($C([0,\infty):\Phi'_p)$ a.s. respectively) and if $\{\phi_j\}_{j \geq 1} \subset \Phi$ is a CONS in Φ_p then

$$\xi_t[\phi] = \gamma[T(0,t)\phi] + \sum_{j=1}^{\infty} \int_0^t \langle T(s,t)\phi, \phi_j\rangle_p dM_s[\phi_j] \quad \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

<u>Proof</u>: It was already noticed at the beginning of the proof of Theorem 2.1 that if q_T does not depend on T neither r_T nor p_T do. Then the theorem holds taking $p > r \geq q$ such that the injection $\Phi_p \hookrightarrow \Phi_q$ is a nuclear operator and given q, r is determined by condition A3(b).

<u>Remark 2.2</u>. The condition on M of Theorem 2.2 can be obtained if there exists ℓ such that for each $t > 0$ there is a $\theta_t > 0$ and $E(M_t[\phi])^2 \leq \theta_t \|\phi\|_\ell^2$ for all $\phi \in \Phi$, for example if $E(M_t[\phi])^2 = tQ(\phi,\phi)$ where $Q(\cdot,\cdot)$ is a positive definite continuous bilinear form on $\Phi \times \Phi$.

<u>Corollary 2.1</u>. Assume γ_0 is a Φ'-valued Gaussian element independent of the Φ'-valued Gaussian martingale with independent increments M_t ($M_0 = 0$) and the family $\{A(t)\}_{t \geq 0}$ satisfies condition (A3) of Theorem 2.1. Then the solution $\xi = (\xi_t)$ of (2.1) given by Theorem 2.1 is a Φ'-valued Gaussian process.

Proof: We only notice that for each $\phi \in \Phi$, from (2.13) we have that $(\xi_t[\phi])_{t\geq 0}$ is a real valued Gaussian process.

Corollary 2.2. Assume (A1)-(A2) in Theorem 2.1 and let A be a continuous linear operator on Φ which is the infinitesimal generator of a $(C_0,1)$-semigroup $\{S(s) : s \geq 0\}$ on Φ. Then the Φ'-valued homogeneous stochastic evolution equation

$$d\xi_t = A'\xi_t dt + dM_t$$
$$\xi_0 = \gamma$$

has a unique solution $\xi = (\xi_t)_{t\geq 0}$ given by the evolution solution

(1) $\xi_t = S'(t)\gamma + \int_0^t S'(t-s)dM_s$

and satisfying (1) in Theorem 2.1. Moreover, ξ is given by

(2) $$\xi_t = M_t + \{S'(t)\gamma + \int_0^t S'(t-s)A'M_s ds\}.$$

If in addition $M \in D([0,\infty) : \Phi_q')$ a.s. (or $C([0,\infty) : \Phi_q')$ a.s.) for some $q > 0$, then there exists $p > q$ such that $\xi \in D([0,\infty) : \Phi_p')$ a.s. (or $C[0,\infty) : \Phi_p')$ a.s.) and if $\{\phi_j\}_{j\geq 1} \subset \Phi$ is a CONS on Φ_p

(3) $$\xi_t[\phi] = \gamma[S(t)\phi] + \sum_{j=1}^{\infty} \int_0^t \langle S(t-s)\phi, \phi_j\rangle_p dM_s[\phi_j] \quad \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

The corollary follows by noticing that a $(C_0,1)$-semigroup on Φ is a $(C_0,1)$-reversed evolution system. In the last statement p should be taken such that $p > \ell > q$ and $\Phi_\ell \to \Phi_p$ is a nuclear operator, and ℓ is such that $|A\phi|_q \leq K|\phi|_\ell$ for all $\phi \in \Phi$ and some constant K.

Finally we consider the solution of the perturbed stochastic evolution equation (2.2).

Theorem 2.3. Assume (A1)-(A3) in Theorem 2.1 and let $\{\|\cdot\|_n : n \geq 0\}$ be the sequence of norms on Φ such that $\{A(t)\}_{t\geq 0}$ is stable with respect to them. Let $\{B(t)\}_{t\geq 0}$ be a family of continuous linear operators on Φ such that there exists $q_0 \geq 0$ and for $q \geq q_0$ and $t \geq 0$ $B(t)$ has a continuous linear extension to $\Phi_{|q|}$ and the map $t \to B(t)$ is $L(\Phi_{|q|}, \Phi_{|q|})$-continuous. Then the Φ'-valued perturbed stochastic evolution equation (2.2) has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ given by the evolution solution

$$\xi_t = V'(t,0)\gamma + \int_0^t V'(t,s)dM_s$$

where $\{V(s,t) : 0 \leq s \leq t < \infty\}$ is the $(C_0,1)$-reversed evolution system generated by the family $\{A(t) + B(t)\}_{t\geq 0}$ given by Theorem 1.4. In addition to (1) in Theorem 2.1 ξ has the following properties:

(a) ξ is given by

$$\xi_t = M_t + \{V'(t,0)\gamma + \int_0^t V'(t,s)(A'(s) + B'(s))\, M_s ds\}.$$

(b) ξ satisfies the integral equation

$$\xi_t = \int_0^t T'(t,s)B'(s)\xi_s ds + \eta_t \quad t \geq 0 \text{ a.s.}$$

i.e.

$$\xi_t[\phi] = \int_0^t \xi_s[B(s)T(s,t)\phi]ds + \eta_t[\phi] \quad \text{for all } \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

where η_t is the unique solution of the unperturbed stochastic equation

$$d\eta_t = A'(t)\eta_t dt + dM_t$$

$$\eta_0 = \gamma$$

and $\{T(s,t) : 0 \leq s \leq t < \infty\}$ is the $(C_0,1)$-reversed evolution system on Φ generated by $\{A(t)\}_{t\geq 0}$.

Proof: Under the conditions on the family $\{B(t)\}_{t\geq 0}$ and using Proposition 1.4 the family $\{A(t) + B(t)\}_{t\geq 0}$ satisfies (A3) in Theorem 2.1. Theorem 1.4 gives the existence of the $(C_0,1)$-reversed evolution system $\{V(s,t) : 0 \leq s \leq t < \infty\}$ on Φ. Then Theorem 2.1 gives the first part of the theorem and (a). Finally using (1.29) in the evolution solution we obtain (b).

Remark 2.3. The main result of this section, Theorem 2.1, has been proved for Φ'-valued martingales such that $E(M_t[\phi])^2 < \infty$ for all $t \geq 0$ and $\phi \in \Phi$. We have been able to relax the requirement of square integrability and show that the stochastic evolution equation (2.1) has a unique solution. The details of the proof as well as the definition of the corresponding stochastic integral will appear elsewhere.

3. EXAMPLES

In this section we consider special cases and examples of stochastic evolution equations and stable families of operators on countably Hilbertian nuclear spaces.

The first two examples illustrate two important facts. First, they are instances where the original problem is given on a Hilbert space H and an appropriate countably Hilbertian nuclear space Φ can be constructed where the problem is solved in a suitable way; secondly, they are examples where the family of operators $\{A(t)\}_{t\geq 0}$ is stable with respect to the sequence of τ-compatible Hilbertian norms on Φ and $A(t)$ is of the form $A + B(t)$ as in Corollary 1.1. These two examples fall within the following framework: Let $(H, \langle\cdot,\cdot\rangle_H)$ be a real separable Hilbert space and $-L$ a closed densely defined self-adjoint operator on H such that $\langle -L\phi, \phi\rangle_H \geq 0$ for each $\phi \in \mathcal{D}(L)$. Let $S(s)$ $s \geq 0$ be the C_0-contraction semigroup on H generated by $-L$. Furthermore assume that some power of the resolvent of L is a Hilbert-Schmidt operator on H, i.e.

$$(3.1) \qquad \exists \ r_1 \text{ such that } (\lambda I + L)^{-r_1} \text{ is Hilbert-Schmidt.}$$

The following construction of a countably Hilbertian nuclear space Φ is well known (see [6]): Condition (3.1) implies that there is a complete orthonormal system $\{\phi_j\}_{j\geq 1}$ in H and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ such that

$$(3.2) \qquad L\phi_j = \lambda_j\phi_j.$$

Define

$$(3.3)\quad \Phi = \{\phi \in H : \|(I+L)^r\phi\|_H^2 < \infty \text{ for all } r \in \mathbb{R}\}$$

$$= \{\phi \in H : \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} \langle \phi, \phi_j \rangle_H^2 < \infty \text{ for all } r \in \mathbb{R}\}.$$

$$(3.4)\quad \langle \phi, \psi \rangle_r = \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} \langle \phi, \phi_j \rangle_H \langle \psi, \phi_j \rangle_H \quad \text{for all } r \in \mathbb{R},\ \phi, \psi \in \Phi.$$

$$(3.5)\quad |\phi|_r^2 = \langle \phi, \phi \rangle_r \quad r \in \mathbb{R},\ \phi \in \Phi.$$

Let Φ_r be the $|\cdot|_r$-completion of Φ. The following three facts are easily verified (see [6]):

(a) The locally convex topology on Φ induced by $|\cdot|_r$ is also given by a countable sequence of norms $|\cdot|_n \geq 0$ and $(\Phi; |\cdot|_n\ n = 0,1,2,\ldots)$ is a countably Hilbertian nuclear space.

(b) For each $r \in \mathbb{R}$

$$(3.6)\quad |S(s)\phi|_r \leq |\phi|_r \quad \text{for all } \phi \in \Phi,\ s \geq 0.$$

Then $S(s) \in L(\Phi,\Phi)$ and extends to a strongly continuous contraction semigroup on each Φ_r:

(c) $\Phi \subset \mathcal{D}(-L)$, $-L\Phi \subseteq \Phi$ and

$$(3.7)\quad |-L\phi|_r \leq |\phi|_{r+1} \quad \text{for all } \phi \in \Phi,\ r \in \mathbb{R}.$$

Hence denoting the restriction of $-L$ to Φ by A, we have $A \in L(\Phi,\Phi)$.

By (3.6) and Theorem 1.1 the restriction of $S(s)$ to Φ, also denoted by $S(s)$, is a $(C_0,1)$-semigroup on Φ. We now prove that A is the infinitesimal generator of $S(s)$ on Φ. Observe first that

$$A\phi = -\sum_{j=1}^{\infty} \lambda_j <\phi,\phi_j>_H \phi_j \qquad \text{for all } \phi \in \Phi$$

$$S(s)\phi = \sum_{j=1}^{\infty} e^{-s\lambda_j} <\phi,\phi_j>_H \phi_j \qquad \text{for all } \phi \in \Phi,\ s \geq 0.$$

Using (3.4) and the last two expressions we have that for $\phi \in \Phi$, $r \in \mathbb{R}$ and $s \geq 0$

$$\|A\phi - \frac{1}{s}(S(s)\phi - \phi)\|_r^2 = \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} <A\phi - \frac{1}{s}(S(s)\phi - \phi),\phi_j>_H^2$$

$$= \sum_{j=1}^{\infty} (1+\lambda_j)^{2r} <\phi,\phi_j>_H^2 (\lambda_j + \frac{1}{s}(e^{-s\lambda_j} - 1))^2.$$

Next, for $j \geq 1$ and $s \geq 0$, from the easily verified inequality

$$(\lambda_j + \frac{1}{s}(e^{-s\lambda_j} - 1))^2 \leq 4\lambda_j^2 \leq 4(\lambda_j + 1)^2$$

we have for $\phi \in \Phi$, $r \in \mathbb{R}$,

$$\|A\phi - \frac{1}{s}(S(s)\phi - \phi)\|_r^2 \leq 4 \sum_{j=1}^{\infty} (1+\lambda_j)^{2(r+1)} <\phi,\phi_j>_H^2 = 4\|\phi\|_{r+1}^2.$$

Thus since $\lambda_j + \frac{1}{s}(e^{-s\lambda_j} - 1) \to 0$ as $s \to 0$, by the dominated convergence theorem,

$$\|A\phi - \frac{1}{s}(S(s)\phi - \phi)\|_r^2 \xrightarrow[s \to 0]{} 0 \qquad \text{for all } \phi \in \Phi,\ r \in \mathbb{R},$$

i.e., A is the infinitesimal generator of the $(C_0,1)$-semigroup $\{S(s): s \geq 0\}$ on Φ. In this case we say that the triple (Φ,H,A) is a *special compatible family*. If in addition there exists a family $\{B(t)\}_{t \geq 0}$ of densely defined linear operators on H such that for all $t \geq 0$, $B(t)\Phi \subset \Phi$ and $\{B(t)\}_{t\geq 0}$ satisfies the assumptions of Corollary 1.1 with respect to the Hilbertian norms $\|\cdot\|_n$, by Corollary 1.1 the family $\{A+B(t)\}_{t\geq 0}$ is stable on Φ and generates a $(C_0,1)$-reversed evolution system satisfying the integral equation (1.31).

Example 3.1. (Christensen and Kallianpur [2], Kallianpur and Wolpert [7])

Let $\Phi \hookrightarrow H \hookrightarrow \Phi'$ be a rigged Hilbert space on which is defined a continuous linear operator $A : \Phi \to \Phi$ and a strongly continuous semigroup $\{S(s) : s \geq 0\}$ on the Hilbert space H such that the following conditions hold:

(i) $S(s)\Phi \subseteq \Phi \quad s \geq 0$.

(ii) The restriction $S(s)|_{\Phi} : \Phi \to \Phi$ is Φ continuous for all $s \geq 0$.

(iii) $s \to S(s)\phi$ is Φ-continuous for all $\phi \in \Phi$.

(iv) The generator $-L$ of $S(s)$ on H coincides with A on Φ.

A semigroup $\{S(s) : s \geq 0\}$ satisfying the above conditions is called compatible with (Φ,H,Φ') or equivalently we say that $(\Phi,H,S(s))$ is a *compatible family* (see [6]).

Consider the stochastic differential equation

$$(3.8) \qquad d\xi_t = -L'\xi_t dt + B'(t)\,\xi_t dt + dM_t$$
$$\xi_0 = \gamma .$$

The unperturbed equation, i.e. $B(t)=0$ $t \geq 0$, is a model used in neurophysiological applications by Christensen and Kallianpur [2] and Kallianpur and Wolpert [7]. The last named authors have solved

(3.8) for the case of a special compatible family, $B(t)=0$ $t\geq 0$ and when M is a Φ'-valued stochastic process with independent increments defined through a Poisson random measure, namely

$$(3.9)\qquad M_t[\phi] = \int_0^t \int_{\mathbb{R}\times X} a\phi(x)\tilde{N}(dadxds) \qquad \phi\in\Phi$$

where $\tilde{N}(dadxds)$ is a compensated Poisson random measure with variance $\mu(dadx)ds$, for some σ-finite measure μ on $\mathbb{R}\times X$. In [7] it is shown that when M is as in (3.9) or a Φ'-valued Wiener process, both M_t and the solution of (3.8) belong to the space $D(\mathbb{R}_+:\Phi'_q)$ a.s. (or $C(\mathbb{R}_+,\Phi'_q)$ in the Wiener process case) where q is independent of t. This is a special case of Corollary 2.2. Example 3.3 in [13] and Example 2.3 in [6] show that we cannot expect a solution lying in $C(\mathbb{R}_+;\Phi'_q)$ a.s. for q independent of t.

In the case of a compatible family and when M_t is a Φ'- valued martingale with $E(M_t[\phi])^2<\infty$ for all $\phi\in\Phi$, $t\geq 0$, the stochastic evolution equation (3.8) with $B(t)=0$ $t\geq 0$ has been solved in Christensen and Kallianpur [2]. Their result is Corollary 2.2 if $S(s)$ is a $(C_0,1)$-semigroup on Φ.

It is important to observe that in neurophysiology the kind of perturbations that occur are more likely to be nonlinear rather than linear.

Example 3.2. (Kotelenez [8])

The stochastic evolution equation of this example has been considered by Kotelenez [8] in connection with fluctuations near homogeneous states of chemical reactions and Gaussian approximation to nonlinear reaction diffusion equations.

Let $C = \{\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1 \quad i = 1, \ldots, n\}$ and let $R(x) = \Sigma_{j=0}^{m} c_j x^j$ be a polynomial in $x \in \mathbb{R}$ where $C_0 \ge 0$ and $C_j < 0$ $j \ge 2$. Consider the nonstochastic partial differential equation

$$(3.10) \qquad \begin{cases} \frac{\partial}{\partial t} X(t,\underline{x}) = D\Delta X(t,\underline{x}) + R(X(t,\underline{x})) \\ X(t,\underline{x}) = 0 \text{ if } x_i = 0 \text{ or } 1 \\ X(0,\underline{x}) \ge 0 \end{cases}$$

where Δ denotes the Laplacian operator and $D > 0$ is a diffusion coefficient. The solution of (3.10) is the concentration of one reactant with reflection at the boundary (see [8]).

Let $H_0 = L_2(C)$ be the real separable Hilbert space of square integrable functions on C with inner product

$$\langle \phi, \psi \rangle_0 = \int_C \phi(\underline{x}) \psi(\underline{x}) d\underline{x} \, .$$

Let A denote the closure of $D\Delta$ in H_0 with respect to the reflecting boundary condition in (3.10). It is well known that A is a self-adjoint dissipative operator on H. Moreover if

$$(3.11) \qquad q_{\ell_i}(x_i) = \begin{cases} \sqrt{2}\cos(\ell_i \pi x_i) & \ell_i \ge 1 \\ 1 & \ell_i = 0 \end{cases}$$

then $\phi_\ell = \Pi_{i=1}^{n} q_{\ell_i}$ is a complete orthonormal system of eigenvectors of A in H_0 (where $\ell = (\ell_1, \ldots, \ell_n)$ is a multiindex) with eigenvalues

$$(3.12) \qquad -\lambda_\ell = -D\pi^2 \sum_{i=1}^{n} \ell_i^2 .$$

Furthermore $\Sigma_\ell (1 + \lambda_\ell)^{-r} < \infty$ for $r > n/2$. Then a countably Hilbertian nuclear space Φ can be constructed as in (3.3) such that for $r \in \mathbb{R}$

$$(3.13) \qquad |\phi|_r^2 = \sum_\ell \langle\phi,\phi_\ell\rangle_0^2 (1+\lambda_\ell)^{2r}$$

and the injection $\Phi_p \hookrightarrow \Phi_q$ is Hilbert-Schmidt for $p > q + r_1$, $r_1 = n/2$. Thus (Φ, H_0, A) is a special compatible family and the restriction of A to Φ (also denoted by A) is the infinitesimal generator of a $(C_0,1)$ semigroup $\{S(s) : s \geq 0\}$ on Φ. The space Φ is the nuclear space of all infinitely differentiable functions $\phi(\underline{x})$ on C such that $\phi(\underline{x})$ and $\frac{\partial^{\ell_1}}{\partial x_1^{\ell_1}} \cdots \frac{\partial^{\ell_n}}{\partial x_n^{\ell_n}}\phi(\underline{x})$ for ℓ_i odd, some i, are zero if x_i is 0 or 1.

Consider the stochastic evolution equation

$$(3.14) \qquad \begin{aligned} d\xi_t &= (A' + B'(t))\,\xi_t dt + dM_t \\ \xi_0 &= \gamma \end{aligned}$$

where γ is an F_0-measurable Φ'-valued gaussian random element independent of the Φ'-valued gaussian martingale $M = (M_t)_{t \geq 0}$ with covariance functional

$$(3.15) \qquad E(M_t[\phi]M_s[\phi]) = \int_0^{t \wedge s} \langle -2D \sum_{i=1}^n \partial_i X(u)\partial_i \phi + \sum_{j=0}^m |c_j| X(u)^j \phi, \phi\rangle_0 du,$$

$B(t) = R^{(1)}(X(t))$ for $X(t) = X(t,\underline{x})$ the solution of (3.10) and $R^{(1)}(x)$ denotes the derivative of $R(x)$, $x \in \mathbb{R}$. $R^{(1)}(X(t))$ acts as a multiplication operator on H_0, i.e.

$$(B(t)f)(\underline{x}) = R^{(1)}(X(t,\underline{x}))f(\underline{x}) \quad f \in H_0.$$

<u>Theorem 3.1</u>. Assume the initial value $X(0,\underline{x})$ of (3.10) satisfies the following conditions:

(i) $0 \leq X(0,\underline{x}) \leq \hat{\partial}$ where $\hat{\partial}$ is some positive number such that $R(x) < 0$ for all $x > \hat{\partial}$

(ii) $X(0,\underline{x})$ is an infinitely differentiable function in $\underline{x}$, with bounded derivatives of all orders which vanish if $x_i = 0$ or 1.

Then the stochastic evolution equation (3.14) has a unique Φ'-valued solution $\xi = (\xi_t)_{t\geq 0}$ which is a Φ'-valued Gaussian process given by the evolution solution

$$\xi_t = T'(t,0)\gamma + \int_0^t T'(t,s)dM_s$$

and satisfying (1) and (2) in Theorem 2.1, where $\{T(s,t) : 0 \leq s \leq t < \infty\}$ is the $(C_0,1)$ reversed evolution system generated by the family $\{A(t) = A + R^{(1)}(X(t))\}_{t\geq 0}$. Moreover, for each $p > n/2 + 1$ $\xi_{\cdot} \in C([0,\infty);\Phi_p')$ a.s. and

$$\xi_t[\phi] = \gamma[T(t,0)\phi] + \sum_{\ell}(1+\lambda_\ell)^{-2p} \int_0^t \langle T(t,s)\phi,\phi_\ell\rangle_p dM_s[\phi_\ell] \quad \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

Proof: We shall verify that the conditions of Theorem 2.2 are satisfied. Since (Φ,H_0,A) is a special compatible family then $A(t) = A$, $t \geq 0$ is a continuous linear operator on Φ generating a $(C_0,1)$-semigroup $\{S(s) : s \geq 0\}$ on Φ. Then we only have to check that the conditions of Corollary 1.1 are satisfied by the family $\{B(t)\}_{t\geq 0}$.

Using conditions (i) and (ii) above, by Theorems A1 and A3 in [8] we have that the solution $X(t,\underline{x})$ of (3.10) is a continuous function in t, infinitely differentiable in $\underline{x}$ with derivatives in $\underline{x}$ continuous in t and

(3.16) $\quad 0 \leq X(t,\underline{x}) \leq \hat{\delta} \quad \underline{x} \in \mathcal{C} \quad t \geq 0.$

Next, it is shown in Lemma A.4 in [8] that for each $q \geq 0$ the norm $|\cdot|_q$ defined in (3.13) is equivalent to the norm

$$\|\phi\|_q = \sum_{0 \leq |\ell| \leq q} \int_C (\partial^\ell \phi(x))^2 d\underline{x} \qquad \phi \in C^\infty(S)$$

where $|\ell| = \ell_1 + \ldots + \ell_n$ and $\partial^\ell = \frac{\partial}{\partial x_{\ell_1}} \cdots \frac{\partial}{\partial x_{\ell_n}}$. Then for each $q \geq 0$ there exists a constant a_q such that if $\phi \in \Phi$ and $t \geq 0$

$$|B(t)\phi|_q^2 \leq a_q \|B(t)\phi\|_q^2 = a_q \sum_{0 \leq |\ell| \leq q} \int_C (\partial^\ell (R^{(1)}(X(t,\underline{x}))\phi(\underline{x}))^2 d\underline{x}$$

and using the Leibniz formula and Schwarz inequality we have that for some positive constant d_q

$$(3.17) \qquad |B(t)\phi|_q^2 \leq d_q \sum_{0 \leq |\ell| \leq q} \int_C \Big(\sum_{i \leq \ell} (\partial^i R^{(1)}(X(t,\underline{x}))^2\Big)\Big(\sum_{i \leq \ell} (\partial^{\ell-i}\phi(\underline{x}))\big)^2\Big) d\underline{x}.$$

Then using (3.16) and since $R(x)$ is a polynomial in x of degree m with constant coefficients, there exist positive constants $d_i(m,q)$ $i = 1,2,3$ such that for any $t \geq 0$

$$(3.18) \qquad |B(t)\phi|_q^2 \leq d_1(m,q) \sum_{0 \leq |\ell| \leq q} \int_C \sum_{i \leq \ell} (\partial^{\ell-1}\phi(\underline{x}))^2 d\underline{x}$$

$$\leq d_2(m,q)\|\phi\|^2 \leq d_3(m,q)|\phi|_q^2 \qquad \phi \in \Phi$$

i.e. $B(t)$ maps Φ into Φ and can be extended to an element in $L(\Phi_q, \Phi_q)$ for $q \geq 0$.

In a similar way to (3.17) and (3.18) it can be shown that for $t,s \in [0,\infty)$

$$|B(t)\phi - B(s)\phi|_q^2 \leq d_4(m,q)|\phi|_q^2 \sum_{0 \leq |\ell| \leq q} \int_C \sum_{i \leq \ell} (\partial^i (R^{(1)}(X(t,\underline{x})) - R^{(1)}(X(s,\underline{x}))))d\underline{x} \quad ; \quad \phi$$

Then using the continuity in t of the derivatives of $X(t,\underline{x})$, the fact that R' is a polynomial and the dominated convergence theorem we have that the map $t \to B(t)$ is $L(\Phi_q, \Phi_q)$-continuous for all $q \geq 0$. Then the conditions of Corollary 1.1 are satisfied and the existence of the $(C_0,1)$-reversed evolution system $T(s,t)$ generated by the family $\{A + B(t)\}_{t \geq 0}$ is established.

Next since γ and $M = (M_t)_{t \geq 0}$ satisfy (A1) and (A2) respectively in Theorem 2.1 then the first part of the theorem follows as an application of Theorem 2.1.

Finally since $E(M_t[\phi])^2 \leq tK|\phi|_1$, $\phi \in \Phi$, $t \geq 0$ for some positive constant K, the Φ'-valued gaussian martingale M has a version in $C([0,\infty);\Phi'_p)$ a.s. for $p > n/2 + 1$ (see [6]). Then from Theorem 2.2 $\xi_{\cdot} \in C([0,\infty);\Phi'_p)$ a.s. and the last statement of the theorem follows.

$\square$

The following is an example where the family of linear operators $\{A(t)\}_{t \geq 0}$ is not of the form A+B(t) nor a special compatible family.

Example 3.3 (Interacting diffusions).

The nuclear space valued stochastic evolution equation of this example occurs as the fluctuation limit for interacting diffusions and it is a perturbed equation of the type (2.2). The study of the limit of interacting particles has been done by McKean [11], Hitsuda and Mitoma [4] and Mitoma [14] amongst others.

For $n \geq 1$ let $Y^{(n)}(t) = (Y_1^{(n)}(t), \ldots, Y_n^{(n)}(t))$ be an n-particle diffusion given by the real valued stochastic differential equation

$$Y_k^{(n)}(t) = \gamma_k + \frac{1}{n} \sum_{j=1}^{n} \int_0^t a(Y_k^{(n)}(s), Y_j^{(n)}(s)) dW_s^k$$

$$+ \frac{1}{n} \sum_{j=1} \int_0^t b(Y_k^{(n)}(s), Y_j^{(n)}(s)) ds \qquad (k = 1, \ldots, n)$$

where $(\gamma_k, W^k)_{k\geq 1}$ are independent copies of (γ, W) and γ is a random variable such that $E(e^{c_0 \gamma}) < \infty$, for some $c_0 > 0$, and independent of the real valued Brownian motion $W = (W_t)_{t\geq 0}$. The coefficients $a(x,y)$ and $b(x,y)$ are bounded C^∞-functions in (x,y) with derivatives in x bounded in (x,y).

Consider the measure valued process

$$U^{(n)}(t) = \frac{1}{n} \sum_{j=1}^{n} \delta_{Y_j^{(n)}(t)} \qquad t \geq 0$$

where δ_x is the unit mass at x. McKean [11] has shown that for each $t \geq 0$

$$U^{(n)}(t) \rightarrow U(\cdot, t) \qquad \text{(in probability)}$$

where $U(dx, t)$ is the probability distribution of Z_t that satisfies the real valued stochatic differential equation

$$dZ_t = \alpha(Z_t, t) dW_t + \beta(Z_t, t) dt$$

and

$$\alpha(x,t) := \int_{\mathbb{R}} a(x,y) U(dy,t)$$

$$\beta(x,t) := \int_{\mathbb{R}} b(x,y) U(dy,t).$$

Moreover, it is also shown in [11] that $U(x,t)$ has a density $u(x,t)$ and that $\alpha(x,t)$, $\beta(x,t)$ and $u(x,t)$ are C^∞-functions on $\mathbb{R} \times \mathbb{R}_+$.

Let

$$S_n(t) = n^{1/2} (U^{(n)}(t) - U(\cdot, t)).$$

Hitsuda and Mitoma[4] have shown that the measure valued processes $S_n(t)$ converge weakly to the solution $\xi = (\xi_t)_{t\geq 0}$ of the nuclear space valued stochastic evolution equation

(3.19)
$$d\xi_t = A'(t)\xi_t dt + B'(t)\xi_t dt + dM_t$$
$$\xi_0 = \gamma$$

where for Φ as defined below

(3.20)
$$(A(t)\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi^{(2)}(x) + \beta(x,t)\phi^{(1)}(x) \quad \text{and}$$

(3.21)
$$(B(t)\phi)(x) = \int_{\mathbb{R}} b(y,x)\phi^{(1)}(y)u(y,t)dy$$
$$+ \int_{\mathbb{R}} \alpha(y,t)a(x,y)\phi^{(2)}(y)u(y,t)dy \quad \text{for } \phi \in \Phi,$$

M_t $(M_0=0)$ is a zero mean Φ'-valued continuous Gaussian martingale with covariance functional

(3.22)
$$E(M_t[\phi_1]M_s[\phi_2]) = \int_0^{t\wedge s}\int_{\mathbb{R}} \phi_1^{(1)}(x)\ \phi_2^{(1)}(x)\alpha(x,r)^2u(dx,r)dr \qquad \phi_1,\phi_2 \in \Phi$$

and γ is a Φ'-valued zero mean Gaussian random element independent of M. As pointed out in [4] the nuclear space appropriate to the problem is given by the space Φ of real valued functions ϕ such that $\phi \in \Phi$ if and only if $\psi\phi \in \mathcal{S}$. (the space of rapidly decreasing functions on $\mathbb{R}$), where

$$\psi(x) = \int_{\mathbb{R}} e^{-|z|}\ \rho(x-z)dz$$

and ρ is the usual mollifier

$$\rho(x) = \begin{cases} c \quad \exp(1/(1-|x|^2)) & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

Observe that Φ is a modification of $\mathcal{S}$ with the following relations among the norms defining their corresponding topologies:

(3.23)
$$||\phi||_n = ||\psi\phi||_{n,\mathcal{S}}$$

(3.24)
$$|\phi|_n = |\psi\phi|_{n,\mathcal{S}}$$

where for $f \in \mathcal{S}$

$$(3.25) \qquad \|f\|_{n,\mathcal{S}} = \sup_{0 \le k \le n} \ \sup_{x \in \mathbb{R}} \ (1+x^2)^n |D^k f(x)| \qquad n \ge 0$$

$$(3.26) \qquad |f|^2_{n,\mathcal{S}} = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{2n} \ |D^k f(x)|^2 \, dx \qquad n \ge 0.$$

From the well known relation between the norms on $\mathcal{S}$, we have that for all $n \ge 1$ there exist constants c_n and d_n such that for every $\phi \in \Phi$

$$c_n \|\phi\|_{n-1} \le |\phi|_n \le d_n \|\phi\|_{n+1}.$$

Some brief comments on the relationship of our treatment of Example 3 with that of Mitoma [14] are in order. In [14] I. Mitoma has found, in our terminology, the $(C_0,1)$-reversed evolution system $T(s,t)$ on Φ generated by the family $\{A(t)\}_{t \ge 0}$ given by (3.19). His main tools are several results of Kunita [10] for stochastic flows of diffeomorphisms of $\mathbb{R}$, including Ito's forward and backward formulas. He then proves, by the method of successive approximations, that the stochastic evolution equation (3.19) has a unique solution. An explicit expression for the solution is not given, nor is it written as an evolution solution. In the theorem below we will prove the existence of the $(C_0,1)$ reversed evolution system $T(s,t)$ on Φ generated by $\{A(t)\}_{t \ge 0}$, by using our Theorem 1.3 and regular Itô's stochastic differential tools. In doing this we show that $\{A(t)\}_{t \ge 0}$ is a stable family of infinitesimal generators of $(C_0,1)$-semigroups on Φ. We then use the perturbation Theorem 1.4 to find the $(C_0,1)$-reversed evolution system $V(s,t)$ generated by the family $\{A(t)+B(t)\}_{t \ge 0}$ and using Theorem 2.2 we are able to write the unique solution of (3.19) as an evolution solution.

<u>Theorem 3.2</u>. Under the above conditions on $a(x,y)$, $b(x,y)$, [illegible] and $M = (M_t)_{t \ge 0}$ there exists a unique $(C_0,1)$-reversed evolution system $\{V(s,t): 0 \le s \le t < \infty\}$ on Φ such that the stochastic evolution equation (3.19) has a unique Φ'-valued

solution which is a Φ'-valued Gaussian [illegible] given by the evolution solution

$$\xi_t = V'(t,0)\cdot + \int_0^t V'(t,s)dM_s .$$

Moreover for any $p>6$ $\xi \in C([0,\infty):\Phi_p')$ a.s. and for $t\geq 0$

$$(3.27)\quad \xi_t[\phi]=\cdot[V(0,t)\phi] = \sum_{j=1}^{\infty} \int_0^t \langle V(s,t)\phi,\phi_j\rangle_p \, dM_s[\phi_j] \quad \text{for all } \phi \in \Phi$$

where $\{\phi_j\}_{j\geq 1} \subset \Phi$ is a complete orthonormal set in Φ_p.

In order to prove the above theorem we will verify the conditions of Theorem 2.3. We first prove that the family $\{A(t)\}_{t\geq 0}$ is stable on Φ.

Proposition 3.1. The family of operators $\{A(t)\}_{t\geq 0}$ defined by (3.20) is uniformly stable on Φ with respect to the norms given by (3.23).

Proof. We first show that for each $t\geq 0$ $A(t)$ maps Φ into Φ and $A(t) \in \mathcal{L}(\Phi,\Phi)$: Let $\phi\in\Phi$ and $n\geq 0$, then from (3.20) we have that

$$D^k(\psi A(t)\phi)(x) = \frac{1}{2}D^k(\psi(x)\alpha(x,t)^2\phi^{(2)}(x)) + D^k(\psi(x)\beta(x,t)\phi^{(1)}(x)).$$

Then since $\alpha(x,t)$ and $\beta(x,t)$ are $C^{\infty}(\mathbb{R}\times\mathbb{R}_+)$ functions with bounded derivatives in x of all order, for each $T>0$ there exist constants $K_1(n,T)$ $i=1,2$ such that for $0\leq k\leq n$ and $0\leq t\leq T$

$$(3.28)\quad |D^k(\psi A(t)\phi)(x)| \leq K_1(n,T) \sum_{i=0}^{k} |D^i\psi(x)\phi^{(2)}(x)| + K_2(n,T) \sum_{i=0}^{k} |D^i\psi(x)\phi^{(1)}(x)| .$$

Next it is not difficult to show that for each $\ell\geq 0$ and $n\geq 0$ there exists a constant $c(\ell,n)$ such that

$$||\phi^{(\ell)}||_n \leq c(\ell,n)||\phi||_{n+\ell} \quad \text{for all } \phi \in \Phi.$$

Then using (3.23) and (3.25) we have that for each $T>0$ and $n\geq 0$ there exists a constant $K_3(n,T)$ such that

$$(3.29)\qquad \sup_{0\leq t\leq T} ||A(t)\phi||_n \leq K_3(n,T)||\phi||_{n+2} \quad \text{for all } \phi\in\Phi$$

which implies $A(t) \in \mathcal{L}(\Phi,\Phi)$ $t\geq 0$.

Next let $t \geq 0$ be fixed but otherwise arbitrary and for $\phi \in \Phi$ and $x \in \mathbb{R}$ define

$$(S_t(s)\phi)(x) = E[\phi(X_s^t(x))] \qquad s \geq 0 \tag{3.30}$$

where

$$X_s^t(x) = x + \int_0^s \alpha(X_r^t(x), t)d\mathscr{B}_r + \int_0^s \beta(X_r^t(x), t)dr \tag{3.31}$$

and $\{\mathscr{B}_r\}_{r \geq 0}$ is a one dimensional Brownian motion. Observe that since $\alpha(x,t)$ and $\beta(x,t)$ are bounded C^∞-functions in $\mathbb{R} \times \mathbb{R}_+$ then (3.31) has a unique solution. Also since for some constant $K > 0$

$$|\alpha(x,t)| \leq K, \quad |\beta(x,t)| \leq K \quad x \in \mathbb{R}, \ t \geq 0, \tag{3.32}$$

using the fact that $|\psi^{-1}(x)| \leq de^{|x|}$ $x \in \mathbb{R}$ (some constant $d > 0$) and Lemma 5.7.1 in [5] we have that for $\phi \in \Phi$

$$E|\phi(X_s^t(x))| = E|\psi^{-1}(X_s^t(x))\psi(X_s^t(x))\phi(X_s^t(x))|$$

$$\leq d||\psi\phi||_{0,\mathscr{S}} \, Ee^{|X_s^t(x)|} < \infty \qquad s \geq 0, \ x \in \mathbb{R}, \ t \geq 0,$$

i.e., (3.30) is well defined. Moreover $S_t(s)\phi$ is linear in Φ and satisfies the semigroup property $S_t(s_1 + s_2) = S_t(s_1)S_t(s_2)$.

Next applying Ito's formula to (3.31) we have that for $\phi \in \Phi$

$$\phi(X_s^t(x)) = \phi(x) + \int_0^s \phi^{(1)}(X_r^t(x))\alpha(X_r^t(x), t)d\mathscr{B}_r + \int_0^s (A(t)\phi)(X_r^t(x))dr$$

and taking expectations in both sides of the last expression and using (3.30) we obtain that for $\phi \in \Phi$ and $s \geq 0$

$$(S_t(s)\phi)(x) = \phi(x) + \int_0^s (S_t(r)A(t)\phi)(x)dr. \tag{3.33}$$

Next we shall prove that $S_t(s)$ maps Φ into Φ and that it is a $(C_0,1)$-semigroup on Φ with infinitesimal generator $A(t)$.

Using (3.23) and (3.25) we have that for $\phi \in \Phi$ and $n \geq 0$

$$||S_t(s)\phi||_n = \sup_{0 \leq m \leq n} \ \sup_{x \in \mathbb{R}} (1+x^2)^n |D^m(\psi S_t(s)\phi)(x)| \tag{3.34}$$

Applying Leibniz formula we have that for $0 \leq m \leq n$ there exists a constant $e(n) \geq 0$ such that

$$|D^m(\psi S_t(s)\phi)(x)| \leq e(n) \sum_{r=0}^{m} |D^r\psi(x)||D^{m-r}(S_t(s)\phi)(x)|.$$

and it is not difficult to show that there are positive constants c(n) and d(n) such that

(3.35) $$|D^r\psi(x)| \leq c(n)e^{-|x|} \quad x\in\mathbb{R},\quad 0\leq r\leq n$$

and

(3.36) $$|D^r\psi^{-1}(x)| \leq d(n)\ e^{|x|} \quad x\in\mathbb{R},\quad 0\leq r\leq n.$$

Then for $x\in\mathbb{R}$ and $0\leq m\leq n$

(3.37) $$|D^m(\psi S_t(s)\phi)(x)| \leq e(n)c(n) \sum_{r=0}^{m} |D^{m-r}(S_t(s)\phi)(x)|.$$

Next using again Leibniz formula and Holder inequality we have that for $0\leq k\leq m\leq n$

$$|D^k(S_t(s)\phi)(x)| = |E(D^k(\psi^{-1}\psi\phi)\ (X_s^t(x)))| \leq$$

(3.38) $$e(n) \sum_{r=0}^{k} (E|D^r\psi^{-1}(X_s^t(x))|^2)^{1/2}(E|D^{k-r}(\psi\phi)(X_s^t(x))|^2)^{1/2}.$$

From (3.36) we have that for $0\leq r\leq n$

$$E|D^r\psi^{-1}(X_s^t(x))|^2 \leq d^2(n)Ee^{2|X_s^t(x)|}$$

and from (3.32) and Lemma 5.7.1 in [5] we obtain

$$Ee^{2|X_s^t(x)|} \leq e^{2|x| + 2sK + 2 + 2K^2 s} \quad s\geq 0,\ t\geq 0,\ x\in\mathbb{R}.$$

Then for $0\leq r\leq n$ and $s\geq 0$

(3.39) $$(E|D^r\psi^{-1}(X_s^t(x))|^2)^{1/2} \leq d(n)e^{|x|+1+(K+2K^2)s}.$$

On the other hand from Lemma 2.3 in [10] there exists a positive constant b(n) such that

$$E\left\{\frac{1}{(1+|X_s^t(x)|^2)^{2n}}\right\}^{1/2} \leq \frac{b(n)s}{(1+x^2)^n} \qquad x\in\mathbb{R},\ s>0.$$

Then using the last inequality and (3.23) and (3.25) we obtain that for $0\leq r\leq n$

(3.40) $$E[|D^r(\psi\phi)(X_s^t(x))|^2]^{1/2} = E[\frac{(1+|X_s^t(x)|^2)^{2n}}{(1+|X_s^t(x)|^2)^{2n}}|D^r(\psi\phi)(X_s^t(x))|^2]^{1/2}$$

$$\leq \frac{b(n)s}{(1+x^2)^n}\ ||\phi||_n \quad \text{for all } \phi\in\Phi,\ s>0.$$

Hence using (3.39) and (3.40) in (3.38) we have that for some constant $f(n)$ and $0 \le k \le n$

$$(3.41)\quad |D^k(S_t(s)\phi)(x)| \le \frac{f(n)}{(1+x^2)^n} e^{|x|} e^{(K+2K^2)s} ||\phi||_n \quad \text{for all } \phi\in\Phi,\ x\in\mathbb{R},\ s\ge 0.$$

Next using (3.41) in (3.37) we have that for some positive constant $g(n)$ and $0 \le m \le n$

$$(3.42)\quad |D^m(\psi S_t(s)\phi)(x)| \le \frac{g(n)s}{(1+x^2)^n} e^{(K+2K^2)s} ||\phi||_n \quad \text{for all } \phi\in\Phi,\ x\in\mathbb{R}\ s>0.$$

Then from (3.34) and (3.42) we have that for any $t\ge 0$

$$(3.43)\qquad ||S_t(s)\phi||_n \le e^{\sigma_n s} ||\phi||_n \quad \text{for all } \phi\in\Phi,\ s\ge 0$$

where $\sigma_n = \log(g(n))(K+2K^2)$ is independent of s or t.

We now prove that for each $t \ge 0$, $S_t(s)$, $s \ge 0$ is a C_0-semigroup: Let $s' < s$, then from (3.33) we have that for $\phi \in \Phi$

$$S_t(s)\phi(x) - S_t(s')\phi(x) = \int_{s'}^{s} (S_t(r)A(t)\phi)(x)dr.$$

Hence for any $0 \le m \le n$

$$D^m(\Psi(x)(S_t(s)\phi(x) - S_t(s')\phi(x)) = \int_{s'}^{s} D^m(\Psi S_t(r)A(t)\phi)(x)dr$$

and using (3.42) and (3.29),

$$|D^m(\Psi(x)(S^t(s)\phi(x) - S_t(s')\phi(x)))| \le \frac{g(n)}{(1+x^2)^n} K_3(n,T)||\phi||_{n+2} \int_{s'}^{s} re^{(K+2K^2)r}dr.$$

Then for any $n \ge 0$ and $\phi \in \Phi$

$$||S_t(s)\phi - S_t(s')\phi||_n \le g(n)K_3(n,T)||\phi||_{n+2}\int_{s'}^{s} re^{(K+2K^2)r}dr$$

$$\to 0 \quad \text{as } s' \to s,$$

i.e., $\{S_t(s): s \geq 0\}$ is a C_0-semigroup on Φ and by (3.43) it is a $(C_0,1)$-semigroup on Φ.

We finally prove that for each $t \geq 0$, $A(t)$ is the infinitesimal generator of $\{S_t(s): s \geq 0\}$ on Φ: From (3.33) for $\phi \in \Phi$ and $s \geq 0$

$$S_t(s)\phi - \phi = \int_0^s S_t(r)A(t)\phi dr.$$

Then for each $n \geq 0$, using the continuity of the map $s \to S_t(s)\phi$ in Φ we have that, for each $\phi \in \Phi$,

$$\begin{aligned} \|A(t)\phi - \frac{1}{s}(S_t(s)\phi - \phi)\|_n &= \|A(t)\phi - \frac{1}{s}\int_0^s S_t(r)A(t)\phi dr\|_n \\ &\leq \frac{1}{s}\int_0^s \|S_t(r)A(t)\phi - A(t)\phi\|_n dr \to 0 \end{aligned}$$

as $s \to 0$, i.e.,

$$A(t)\phi = \underset{s \downarrow t}{\text{limit}} \frac{1}{s}(S_t(s)\phi - \phi) \qquad (\text{limit in } \Phi) \quad t \geq 0.$$

Hence $A(t)$ is the infinitesimal generator on Φ of the $(C_0,1)$-semigroup $\{S_t(s): s \geq 0\}$.

Finally from (3.43) and Proposition 1.3 the family $\{A(t)\}_{t\geq 0}$ is a uniformly stable family of continuous linear operators on Φ. The proof of the proposition is complete.

Proof of Theorem 3.2.

Using (3.24) and similar arguments to those in proving (3.28) and (3.29) it is easy to show that for each $T>0$ and $n\geq 0$ there exists $K_4(n,T)>0$ such that for $t,t' \in [0,T]$

(3.44) $$|A(t)\phi - A(t')\phi|_n \leq K_4(n,T)|\phi|_{n+4}\, h_n(t,t') \quad \text{for all } \phi \in \Phi$$

where

$$h_n(t,t') = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{-2} | D^k(\alpha(x,t)-\alpha(x,t'))|^2 dx$$
$$+ \sum_{k=0}^{n} \int_{\mathbb{R}} (1+x^2)^{-2} |D^k(\beta(x,t) - \beta(x,t'))|^2 dx.$$

Also it is not difficult to show that for $n\geq 2$ there exist positive constants $K_5(n,T)$ and $K_6(n,T)$ such that for $t,t' \in [0,T]$ and $\phi\in\Phi$

(3.45) $$|B(t)\phi|_n \leq K_5(n,T)|\phi|_n$$

(3.46) $$|B(t)\phi - B(t')\phi|_n \leq K_6(n,T)|\phi|_n\, g(t,t'),$$

where

$$g(t,t') = \int_{\mathbb{R}} e^y |u(y,t)-u(y,t')|dy$$
$$+\int_{\mathbb{R}} e^y |\alpha(y,t)u(t,y) - \alpha(y,t')u(y,t')|dy.$$

Notice that since all derivatives in x of $\alpha(x,t)$ and $\beta(x,t)$ are bounded in x and continuous in t, by the dominated convergence theorem $h(t,t')\to 0$ as $t'\to t$. Also from Theorem 5.7.2 in [5], for each $T>0$

$$\int_{\mathbb{R}} e^{|y|} u(y,t)dy < \infty \quad 0\leq t\leq T.$$

Then by the dominated convergence theorem $g(t,t') \to 0$ as $t \to t'$.

Next by Proposition 3.1 and (3.44) the family $\{A(t)\}_{t\geq 0}$ satisfies conditions (a) and (b) in Theorem 1.3 and it generates a unique $(C_0,1)$-reversed evolution system $\{T(s,t): 0\leq s\leq t<\infty\}$ on Φ. From (3.45) and (3.46) the family $\{B(t)\}_{t\geq 0}$ satisfies the conditions of Theorem 1.4 and the family $\{A(t)+B(t)\}_{t\geq 0}$ generates a unique $(C_0,1)$-reversed evolution system $\{V(s,t): 0\leq s\leq t<\infty\}$ on Φ. The theorem then follows applying Theorem 2.3.

Finally from (3.22) for each $t>0$ there exists $K_7(t)>0$ such that for $\phi\in\Phi$

$$E(M_t[\phi])^2 \leq K_7(t)|\phi|_2.$$

Then by (3.44) and Theorem 2.2, for any integer $p>6$ we obtain $\xi \in C([0,\infty):\Phi_p')$ a.s. and (3.27). □

In [18] Tanaka and Hitsuda consider a simple diffusion model of interacting particles. The stochastic evolution equation of their example can be solved in the framework of a special compatible family of the form $A+B(t)$, as in Examples 3.1 and 3.2.

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